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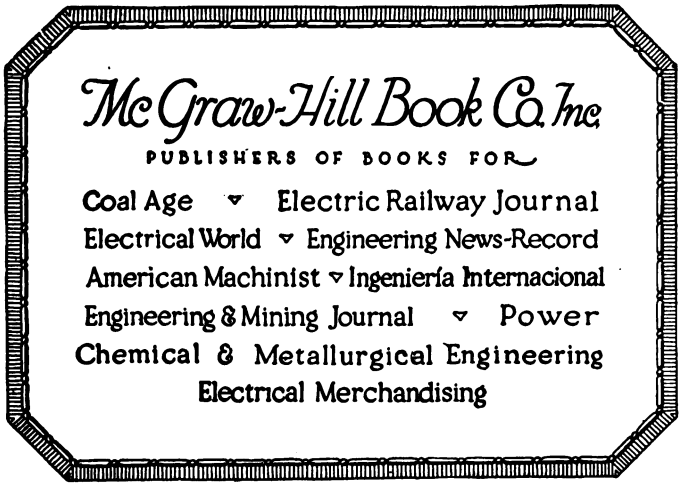




ESSENTIALS

IN THE

THEORY OF FRAMED STRUCTURES


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ESSENTIALS

IN THE

THEORY OF FRAMED STRUCTURES

BY

CHARLES A. ELLIS, A. B.

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PREFACE

This book, designed primarily for class-room use, contains no new principles. The same fundamental ideas to be found in many other books are here set forth by a treatment so different from the orthodox method that a word of explanation seems justifiable.

There is a growing tendency in engineering schools to instruct rather than to educate; to pour in rather than to draw out; to feed the mind with memoranda until it becomes sluggish for lack of exercise in the intellectual realms of the imagination and reason. This tendency is, to a large extent, the result of current text-book and class-room methods which present mathematical principles in generalities first and particulars last. This order is undesirable for two reasons: First, it is not the sequence most natural for the grasp of the human intellect, for the history of any science will show that its development has been accomplished by progress from the particular to the general. Second, if generalities are considered first, the principle is unfolded to the class by deriving a formula. Once developed, this formula is memorized, and all subsequent problems are solved by merely making numerical substitutions for the various symbols. Thus, the student gets but a passing glance at the fundamental idea and a meager conception of the principle involved. This statement can be proved very easily by asking any group of students who can glibly recite:

$$f = \frac{My}{I} \text{ or } S = \frac{Mc}{I}$$

to determine the fiber stress at a certain point in a given beam without using a formula. This experiment was tried in nearly one thousand cases with the result that about 93 per cent of the students were found to be helpless without the formula.

This instance is more fully discussed by the author in an article "Reinforced Concrete Theory Without the Use of Formulas," Bulletin of the Society for the Promotion of Engineering Education, Vol. VIII, p. 380.

In this book each principle is introduced by the use of illustrative numerical problems, some of which will lead the student into a pitfall from which he should be forced to extricate himself without the instructor's help. Experience is a splendid teacher. There is no opportunity for the student to dodge the fundamental idea or the physical conception of the problem by resorting to a formula, for none is given except at the end of the discussion when the general case is considered. Even then the general expression is frequently left to be developed by the student. The experienced teacher who has had considerable practical experience will, no doubt, feel more at home with this method of treatment than will the younger instructor who is closely confined to his text and who feels that a knowledge of the course is of more importance to the student than the development of mental power.

The major portion of the book may be read intelligently by any student who has a working knowledge of arithmetic, algebra and geometry. Very little reference (except in Chapter V) is made to the calculus. Practically every other textbook makes use of formal calculus in the treatment of maximum stresses for moving loads. Of course, this method is entirely unnecessary. It is true that the fundamental concept of the differential calculus is ever present in the treatment of maximum stresses in connection with influence lines as given in Chapter IV; but the average college student who can differentiate and integrate will probably not recognize the calculus, so why risk confusing him by any reference whatever to

$$\frac{dy}{dx}$$

Some new material will be found in Chapter III relative to wind reactions of a mill-binding bent. A treatment of deflections by the area-moment method, more comprehensive than is usually found in American textbooks, appears in Chapter V.

A new short method for computing the stresses in a swing span is given in Chapter VIII.

The author recommends that the teaching schedule be arranged to include one three-hour period a week in addition to the two or three regular recitation periods. These three-hour periods should be used for written tests in which the student is given an opportunity to solve problems which are just ahead of the textbook assignments, or which were considered so far back in the student's course that he has forgotten the solution; the idea being to give the student some opportunity for independent thought. For example: Suppose that during a certain week the textbook assignments are in Sec. III, Chapter IV; and that the stress in a diagonal of a parallel chord truss has been discussed in the recitation period—the student having solved several problems of this character. The following questions are recommended for the three-hour period of that week:

1. Prove that the sum of the three angles of the triangle equals two right angles.

2. Compute the maximum tensile stress in the number U_2L_3 of the curved chord truss (Fig. 117) for an E-60 train load.

Any simple fundamental problem in algebra, calculus or mechanics will serve the purpose equally as well as the problem in geometry. If the student has mastered the principle of maximum stress in the diagonal of a parallel chord truss he should be given the opportunity of doing some independent thinking on the principle next in order, instead of being asked to memorize that principle from the text in preparation for recitation.

The instructor who expects that the student will submit 100 per cent papers on an examination of this sort will be sadly disappointed; and the student who is not accustomed to this method of conducting tests will at first feel that he is being treated rather roughly; but the mental development which results from tests of this character is of more value to the student than high grades.

The major portion of this book has been used for several years as preprints in the author's classes and he is indebted

to former students for many helpful suggestions. The author wishes also to acknowledge his indebtedness to Professor Geo. E. Beggs for permission to use his table of equivalent uniform loads.

CHARLES A. ELLIS.

CHICAGO, ILLINOIS,
Sept. 29, 1921.

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ESSENTIALS

IN THE

THEORY OF FRAMED STRUCTURES

CHAPTER I

EQUILIBRIUM OF COPLANAR FORCES

SEC. I. INTRODUCTION

1. **Mechanics.**—The theory of stresses in framed structures is the development of a few, simple, fundamental principles of mechanics. The science of mechanics is a study of motion. The idea of motion is closely related to the ideas of space, time and mass.

We study first the motion of a body without regard to the time consumed in the motion, or the mass of the thing moved. This part of mechanics is called "Geometry of Motion."

Secondly, the introduction of the idea of time leads us to consider velocity and acceleration. This study is called "Kinematics."

Finally, when the body is assigned to a certain mass, we consider the ideas of energy, momentum and force. This part of mechanics is called "Dynamics" and is divided into two branches—"Kinetics" and "Statics."

Kinetics treats in the most general way of changes in motion produced by forces. Statics is a consideration of those cases in which no change of motion is produced, and the element of time is not a factor.

The fundamental conception of stress analysis in framed structures is acquired by a systematic study of that part of Statics which treats of the "Equilibrium of Coplanar Forces."

This topic is not a new one to the student of Physics and Mechanics. In both of these courses it is treated in a general way in connection with many kindred topics, equally important in other fields of engineering science.

The first chapter of this book has been prepared for the purpose of making an extended investigation of the "Equilibrium of Coplanar Forces," to the end that the reader may acquire a more specific knowledge of the subject and the power to apply this knowledge with reason and common sense. It is the most important chapter in the book; it forms the base upon which rest the principles of each succeeding chapter, and should be thoroughly mastered.

2. Methods and Mental Attitude.—Whenever we engage in a new piece of work, it is wise to consider the tools with which we may accomplish our task, and the manner in which these tools can be used most advantageously. The man who is about to cut down a tree decides, not only whether he will use an axe or a saw, but also (and quite unconsciously, perhaps), whether he will operate the chosen implement with his hands or his feet.

The student will do well to emulate the prudence of the wood cutter and, for the moment, consider the tools or methods by which stresses may be determined. He should also take thought of the much more important question—By what mental processes can these tools most successfully be operated?

There are two methods by which stresses may be determined: (*a*) the algebraic method, which makes use of numerals or symbols; (*b*) the graphic method, which represents quantities by the length and direction of lines.

Graphic methods are not limited to the solution of static problems. The operations of addition, subtraction, multiplication, division, involution and evolution can be performed graphically; and many problems in calculus are more readily comprehended when illustrated by the graphic method. Whether the algebraic or the graphic method is better, is a question which depends for its answer upon the individual problem. However, if each problem is solved by both methods, the fundamental principles are more clearly set before the beginner.

The student should always remember that mental development is more to be desired than the mere accumulation of information, which is of little value unless its possessor has the mental power or capacity for using it intelligently. Since the student can acquire intellectual power only by intellectual exercise, he is frequently asked by his instructor to think. This request may be interpreted in several different ways. One may literally comply with the request if he memorizes, imagines, reasons or performs one of several other minor intellectual functions. The intellectual development, however, varies with the form of exercise—a fact which the instructor should not fail to recognize. Although the training of the memory is an important intellectual process, imagination and reasoning should not be neglected. Which of these three plays the leading role in the drama of life is a question which is answered differently perhaps by the linguist, poet and scientist. It is a significant fact, however, that the majority of people who plead guilty to possessing a poor memory are much offended when accused of having little imagination or inferior judgment.

The degree to which the student will exercise his memory, his imagination or his reason in the preparation of a lesson, depends to a considerable degree upon the manner in which the subject is presented in the textbook. If he sees a formula, his first impulse is to memorize it, too often in parrot-like form; while its history, meaning and limitations are to him a matter for secondary consideration. In a text book or in the class-room, formulas should be given what little place they deserve at the end, and not at the beginning of a subject; for the simple reason that algebraic manipulations obscure the physical relations which should be visualized to the highest degree.

It may appear to the casual reader that imagination is an intellectual process exercised exclusively by the artist or poet. Such is not the case. A structure, whether it be a steam engine, electric generator or a railway bridge must be visualized first in the mind of the engineer before it can become a reality; or even before it can take shape and size upon the designing board.

In preparing the manuscript for this book, the author has

endeavored to present each topic in such form that the student will find employment for his imagination and reason as well as for his memory. Incidentally, he will acquire a working knowledge of the theory of stresses in framed structures.

3. Forces.—The word "force" is used to express a variety of ideas. We speak of physical, mental and moral forces, etc. But for scientific purposes it is necessary to have a single definite meaning. In Physics and Mechanics we define force as the cause of acceleration or change in velocity; *i.e.*

	force	= rate of change of momentum
and	momentum	= mass × velocity
hence	force	= mass × rate of change of velocity
or since	rate of change of velocity	= acceleration
therefore	force	= mass × acceleration.

All bodies are constantly acted upon by the force of gravitation; and, except in the special case of bodies falling in a vacuum, by other forces also. A moving train on a straight, level track is usually subject to the action of four forces: the action of gravitation "pulling down" and the reaction of the rails "pushing up"; the "pull" of the engine in one direction and the retarding action of friction and atmosphere in the opposite direction.

If the reaction of the rails equals the action of gravitation, the train will remain at the same level. If the engine pull is greater than the retarding action, the difference represents the force which accelerates the speed of the train; thereby increasing its velocity and momentum. On the other hand, if the force represented by the engine pull equals the retarding forces, the train is moving at a constant speed, and there is no rate of change of velocity or momentum. The two equal and opposite forces "balance" and their algebraic sum is zero. Thus if a body is at rest, or is moving in a straight line with constant velocity, the acceleration is zero, there is no change in momentum, and the body is said to be in a state of equilibrium.

The theory of stresses in framed structures is developed by a study of the action of forces on bodies at rest. We need give no special attention to velocity, momentum or acceleration in our conception of the idea of force. For our present discussion we may therefore properly conceive of a force as the idea expressed in the words "push" and "pull."

4. The Three Elements of a Force.—We know that the effect of a force of 50 lb. pushing vertically (downward) upon the head is quite different from the effect of a force of 80 lb. pulling horizontally at the feet; and thus from our personal experiences we may observe that the characteristics or elements of a force are:

1. Magnitude.
2. Direction.
3. Location.

The effect of the action of a force upon a body cannot be determined until all three elements are known.

The *magnitude* or intensity of a force is usually expressed in pounds, units of 1,000 lb., or tons.

The *direction* may be indicated by the angle between the horizontal and the line along which the force acts. For our purpose, however, it will be found more advantageous to designate the direction by giving the slope or bevel of the line along which the force acts.

The *location* of a force is known when the position (with reference to the body) of any point in the line representing the direction is given.

The *sense* in which a force acts along a given direction, *i.e.*, up or down, toward the right or left, is represented graphically by an arrow-head. In an algebraic solution the sense is designated by the signs + or − in connection with the magnitude of the force. The sense of a force is not ranked as an element, since it requires no separate equation for its determination.

5. Definitions.—Forces are *coplanar* when acting in the same plane; non-coplanar, when acting in different planes.

Forces are *concurrent* when their directions meet in a point, and *non-concurrent* when their directions do not so meet.

Several forces under consideration are called a *system* of forces.

A *couple* is a system of two parallel forces, equal in magnitude, but opposite in sense.

Two systems of forces are *equivalent* if one system may be substituted for the other without changing the state of rest or motion of the body upon which they act.

The *resolution of forces* is the process of resolving the forces of one system into an equivalent system having a greater number of forces.

The *composition of forces* is the process of combining or reducing the forces of one system into an equivalent system having fewer forces. If the equivalent system reduces to a single force, that force is called the *resultant* of the system; and each force in the system is a *component* of the resultant force. All systems, however, cannot be reduced to a single force; but all coplanar systems not in equilibrium can be reduced to either a single force or a couple.

SEC. II. RESOLUTION AND COMBINATION OF FORCES

6. Parallelogram of Forces.—The law of "Triangle of Forces" seems to have been discovered in the year 1608 by Simon Stevin, a Flemish military engineer, who conceived the idea of the resolution and composition of forces, by his investigation of the properties of the inclined plane. He derived the law from experimental data. Mathematical proofs of this law have frequently been presented; but several eminent writers of the present day seem to be content with the statement that the law is so fundamental that it cannot be deduced from anything more simple than itself.

An experimental demonstration of the law is illustrated in Fig. 1. Three smoothly running pulleys are mounted on a vertical drawing board at any points A, B, and C not in the same straight line. Three strings are tied together, placed over the pulleys and made to support three unequal weights, P, Q, and S. Under the action of gravitation, the weights will adjust themselves and come to a state of rest. Mark on the board the position of the knot at o. Add a small weight to P and note the change in the position of the knot. Remove the small weight and observe that the knot returns to its original position. Do likewise with Q, and with S. Now increase each weight by an equal amount and note that the knot changes its position. Increase each weight by one-half or one-third of itself and observe that the position of the knot is not changed.

Under the action of gravitation, each weight produces a corresponding tension in the string which supports it; and we have a system of three concurrent forces in equilibrium.

In order that a line may be employed to represent a force it is necessary that: (1) its length should indicate the magnitude of the force; (2) its direction, with an arrow-head to show the sense, should correspond to the line of action of the force; (3) the location of a point in the line should be given.

The lines oA, oB and oC with arrow-heads to show the sense, represent the directions of the three forces, and the point o designates their location. By choosing a convenient scale-ratio, let us say, 10 lb. per inch, the lengths oa, ob and oc may be laid off on the lines oA, oB and oC to represent the magnitudes,

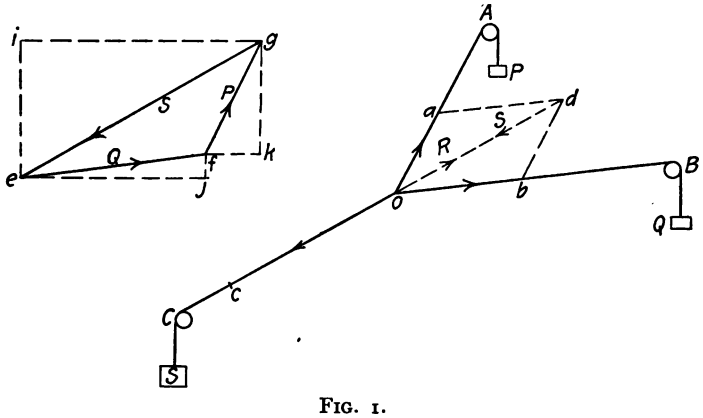


FIG. 1.

P, Q and S respectively. The sequence of the letters which designate the force may be used to indicate the sense of the force. Thus oa signifies that the sense of the force P is upward from o to a, while ao would represent a force the equivalent of P in all respects except that the sense would be opposite, *i.e.*, downward from a to o. The forces P, Q and S are thus completely represented by the lines oa, ob and oc, since each line designates the magnitude, direction (including sense) and location of its respective force.

Draw a line through a, parallel to ob, and a line through b, parallel to oa completing the parallelogram $oadb$. Measure the

length of the diagonal *od*, with the same scale as before. *Observe that the length do equals the length oc; and that the points c, o and d are in a straight line.*

We say that the force *S*, represented by *do*, is the equilibrant of the forces *P* and *Q*, since it holds them in equilibrium or balances them. With equal propriety we can say that a force *R*, represented by *od* having equal magnitude, the same direction and location as the force *S* but opposite sense, would hold the force *S* in equilibrium, if the forces *P* and *Q* were removed. Hence, the force *R* is called the resultant of the forces *P* and *Q*, since it may be substituted for them without disturbing the state of equilibrium. The law of the parallelogram of forces may be stated as follows:

If two concurrent coplanar forces are represented in magnitude and direction by the adjacent sides of a parallelogram, so constructed that the sense of each force is the same with respect to their point of intersection; then the resultant of the two forces is represented in magnitude and direction by the diagonal of the parallelogram through the point of intersection. If the sense of the two forces is away from the point of intersection, the sense of the resultant is away from the point and *vice versa*.

7. Triangle of Forces.—Since a diagonal divides a parallelogram into two equal and similar triangles, it is necessary to construct but one of the triangles in order to find the resultant. Instead of representing the two forces *P* and *Q* by the lines *oa* and *ob* drawn from *o*, we may draw *oa* representing the force *P*; and from *a* draw *ad* representing the force *Q*. Or, we may first draw *ob* to represent the force *Q*, and from *b* draw *bd* representing the force *P*. In either construction the line *od* represents the resultant *R*, of the forces *P* and *Q*; while the line *do* represents their equilibrant *S*.

In a concurrent system, all forces act through a common point which establishes the location of each force, and we shall be concerned chiefly with the determination of unknown magnitudes and directions. For this reason it is not essential that the triangle be constructed in connection with the point of concurrency *o*. The triangle *efg*, for example, having sides respectively parallel to *ob*, *bd* and *do* will serve our purpose in a

consideration of the forces P, Q and S, equally as well as the similar triangle *obd*. The homologous sides of the two triangles are proportional and parallel—an important fact to remember. The triangles differ in size simply because different scale-ratios were used in their construction.

If the sides of a triangle represent three forces in equilibrium, the arrow-heads will indicate a continuous course around the perimeter of the triangle. If, however, the three sides of the triangle represent two forces and their resultant, the arrow-head of the resultant will be opposed to the arrow-heads of the two other forces.

The law of the triangle of forces may be stated as follows: *Three concurrent coplanar forces in equilibrium may be represented in magnitude, direction and sense by the three sides of a triangle taken in continuous order. Any one of the three forces may be made to represent the resultant of the other two by reversing its sense.*

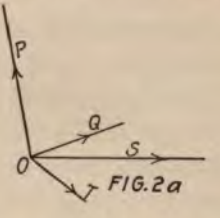


FIG. 2a

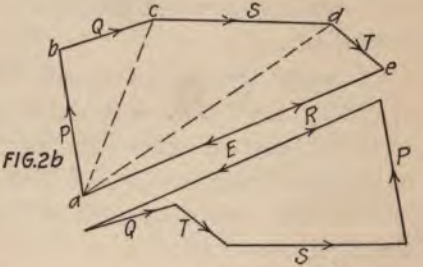


FIG. 2b

FIG. 2c

8. Magnitude-direction Diagram.—When the resultant of several concurrent forces is required, the resultant of any two may be found from the force triangle; and this resultant combined with a third force to obtain the resultant of the three forces. This process may be continued until all forces have been combined. A system of four coplanar, concurrent forces P, Q, S and T intersecting at O is represented in Fig. 2*a*. In Fig. 2*b*, *ac* is the resultant of P and Q; *ad* is the resultant of *ac* and S; and finally *ae*, the resultant of *ad* and T, is the resultant R of the entire system. Evidently the force E, represented by *ea*, having the same magnitude and direction as R but opposite in sense, is the equilibrant which, if applied at O' would secure equilibrium. The partial resultants *ac* and *ad*

are obviously superfluous, since the figure *abcde* could have been drawn without them.

Textbooks in Physics and Mechanics usually refer to the diagram in Fig. 2*b*, as a *force polygon*. The name is misleading. The figure does not completely represent a force—it represents only two of the three essential elements. Magnitude-direction diagram is a more distinctive term. The order in which the forces are combined is of no particular consequence. In Fig. 2*c* the forces have been combined in the order, Q, T, S, P; but the resultant R, or the equilibrant E, are the same as in Fig. 2*b*.

The known magnitudes and directions were laid off from *a* to *e*; and the magnitude and direction of the equilibrant E was found by drawing the line *ea*, which necessarily closed the figure. It follows that a graphic solution for equilibrium must comply with the following law:

If a system of coplanar, concurrent forces is in equilibrium, the magnitude-direction diagram will close, and the sense will be continuous.

Of course the magnitude-direction diagram could have been closed by any series of straight, broken lines, continuous from *e* to *a*; but certain limitations must be imposed upon the number of unknown magnitudes and directions if a single solution is to be realized. We shall refer to this in detail in connection with the algebraic solution.

9. Rectangular Components.—A force may be resolved into two components by applying the triangle law conversely. Since an infinite number of triangles can be drawn having one side in common, it follows that the problem of finding the two components of a given force is indeterminate unless certain conditions are imposed upon the components. Rectangular components have directions intersecting at an angle of 90°, and are very helpful in algebraic solutions; since an inclined force can thus be resolved into and replaced by its horizontal and vertical components. Thus in Fig. 1 the three forces in equilibrium, P, Q and S are resolved into horizontal and vertical components by the right triangles *fkg*, *ejf* and *gie* respectively. The horizontal components *ej* and *fk* are balanced by the component

gi, being equal in magnitude and opposite in sense. Likewise the vertical components *jf* and *kg* are balanced by the component *ie*.

Whenever the magnitude-direction diagram is a closed figure, whether a triangle, as *efg* (Fig. 1); or any polygon, as *abcde* (Fig. 2), the horizontal components will balance and the vertical components will balance. If opposite signs (+ and −) are given to magnitudes of opposite sense, the algebraic sum[1] of the horizontal components is:

$$\Sigma H = 0$$

and likewise

$$\Sigma V = 0$$

These two equations in an algebraic solution correspond to the closing of the magnitude-direction diagram in a graphic solution.

10. Moment of a Force.—The product of the magnitude of a force and its perpendicular distance from a point is called the moment of the force about the point. The perpendicular distance is called the arm of the force, and the point is known as the center of moments. If the magnitude of a force P is 8 lb., and its arm from a point O is 3 ft., the moment of the force P about the point O is 24 ft.-lb.

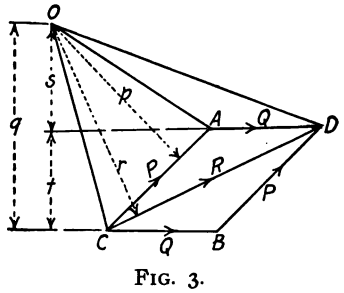


FIG. 3.

In 1687, or about 80 years after Stevin had published his demonstration of the triangles of forces, Pierre Varignon presented before the Paris Academy, the "Principle of Moments." This principle is known as Varignon's Theorem. He stated that "the moment of the diagonal of a parallelogram of forces equals the sum of the moments of the other two sides." His proof is somewhat as follows:

The parallelogram *CADB* (Fig. 3) represents the forces P and Q and their resultant R. The lengths of the perpendiculars from the center of moments O to the lines representing the forces P, Q and R, are p, q and r respectively:

[1] The symbol Σ represents the idea expressed in the words "algebraic sum."

area ΔOCD = area ΔOCA + area ΔOAD + area ΔACD

or $$\tfrac{1}{2} rR = \tfrac{1}{2} pP + \tfrac{1}{2} sQ + \tfrac{1}{2} tQ$$

hence, $$rR = pP + qQ$$

Varignon showed that the center of moments O may be located outside the parallelogram, within it, or on one of its sides.

It is obvious that the moment rR of the resultant R, acting counter-clockwise about the point O, may be balanced by the moment rE of the equilibrant E acting clockwise. If opposite signs are given to the moments of the two forces according as they act clockwise or counter-clockwise, their algebraic sum is zero, or

$$rR - rE = 0$$

therefore, $$pP + qQ - rE = 0$$

P, Q and E represent three coplanar, concurrent forces in equilibrium; but the principles may be generalized so as to include any number of concurrent forces by combining them into partial resultants after the manner of Fig. 2. Hence, if a system of coplanar, concurrent forces is in equilibrium, the algebraic sum of the moments of all the forces about any point in their plane is

$$\Sigma M = 0$$

SEC. III. THE THREE FUNDAMENTAL STATEMENTS OR EQUATIONS OF STATIC EQUILIBRIUM

11. Problems in stress analysis deal in general with the action of coplanar forces upon bodies at rest, and the solution of any problem consists in finding the unknown elements of these forces. If the problem is "statically determinate," the number of unknown elements will not exceed the number of independent statements or equations which can be written concerning the static equilibrium of the body upon which the forces are acting. It is a simple matter to write these equations if we consider carefully the conditions under which a body may be at rest.

If a body is at rest under the action of any system of forces lying in a vertical plane, the following statements are self-evident from the very nature of the case.

1. The body is not moving toward the right or toward the left, consequently the horizontal magnitudes acting toward the right are balanced by the horizontal magnitudes acting toward the left. If the horizontal magnitudes acting toward the right are given a positive sign, and the horizontal magnitudes acting toward the left are given a negative sign, the algebraic sum of the horizontal magnitude is

$$\Sigma H = 0 \qquad (1)$$

2. The body is not moving upward or downward, consequently the vertical magnitudes acting upward are balanced by the vertical magnitudes acting downward. If the vertical magnitudes acting upward are given a positive sign, and the vertical magnitudes acting downward are given a negative sign, the algebraic sum of the vertical magnitude is

$$\Sigma V = 0 \qquad (2)$$

3. The body is not rotating, either clockwise or counter-clockwise, about any point in the plane of the forces; consequently the moments of all the magnitudes acting clockwise about any point in the plane of the forces are balanced by the moments of all the magnitudes acting counter-clockwise about the same point. If clockwise moments are given a positive sign, and counter-clockwise moments are given a negative sign, the algebraic sum of the moments about any point in the plane of the forces is

$$\Sigma M = 0 \qquad (3)$$

These three statements or equations are the conditions which must be fulfilled by any system of forces acting in a vertical plane upon a body at rest, whether the system be concurrent, parallel or non-concurrent non-parallel. Equations (1), (2) and (3) embody the three fundamental principles of static equilibrium for coplanar forces. They are as fundamental and important as they are simple.

Equations (1) and (2) are easily written for any system, after each inclined force has been resolved and replaced by its hori-

zontal and vertical components. Instead of resolving the forces along horizontal and vertical axes, we may choose any other pair of axes inclined at any angle and equate to zero the algebraic sum of the components parallel to each axis. Thus, we may write many equations of types (1) and (2). We may write also many equations of type (3) by choosing different points for the center of moments.

Since a single solution is possible only when the number of unknown elements does not exceed the number of *independent* equations which may be written, the question immediately arises—How many equations, written as above, will be independent for a given system of forces? In answering this question we must distinguish between concurrent, parallel, and non-concurrent non-parallel systems; and for this reason the three systems will be considered separately.

SEC. IV. COPLANAR CONCURRENT FORCES

12. Illustrative Problem.—Five concurrent forces, A, B, C, D and E are located by the point O in Fig. 4*a*. The direction of each force is given, and the magnitudes of D and E are known. We shall attempt to find (*a*) by the algebraic method and (*b*) by the graphic method, the magnitudes of A, B and C necessary for equilibrium.

(*a*) *Algebraic Method.*—We first resolve each inclined force into horizontal and vertical components, and indicate them in the sketch (Fig. 4*b*).

The magnitude of the force D is 100 lb. It acts to the left and upward, sloping at the bevel of 3 in. horizontally to 4 in. vertically; as indicated by the arrow-head at D and the sides fj and jk of the triangle fjk. Hence,

$$fj : jk : fk :: 3 : 4 : 5$$

Let $D = 100$ lb. represent the magnitude of the force,

D_h represent the horizontal component,

and D_v represent the vertical component.

Then, if fjk is considered as a force triangle,

$$D_h : D_v : D :: fj : jk : fk$$

or $D_h : D_v :: 100 :: 3 \text{ in.} : 4 \text{ in.} : 5 \text{ in.}$

$D_h = 60$ lb., acting to the left

and $D_v = 80$ lb., acting upward.

The magnitude and sense of the force A are unknown. For the present we shall assume (or guess) that the sense is away from the point O, and let

$A_h = 8a.$, acting to the right,

then $A_v = 15a.$, acting upward;

and $A = a\sqrt{8^2 + 15^2} = 17a.$

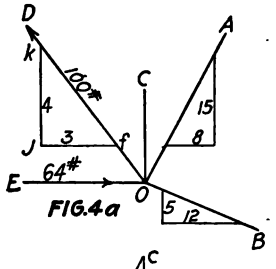


FIG.4a

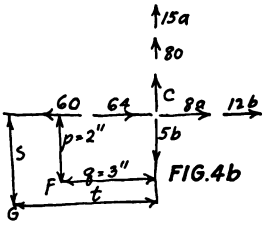


FIG.4b

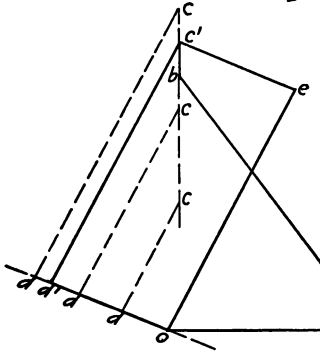


FIG.4c

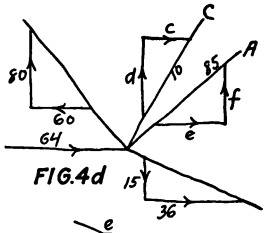


FIG.4d

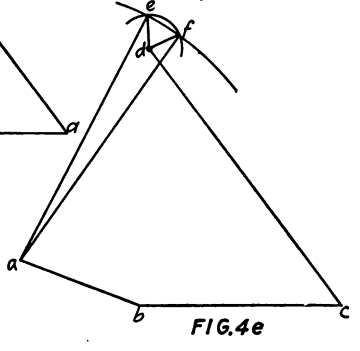


FIG.4e

Assume that the force B acts away from O, and let

$B_h = 12b$, acting to the right,

$B_v = 5b$, acting downward,

and $B = b\sqrt{12^2 + 5^2} = 13b.$

The forces E and C (C is assumed to act upward) and the components of the inclined forces D, A and B are shown in the sketch (Fig. 4*b*). Three independent equations are required for a solution of the three unknown quantities C, a and b.

$$\Sigma H = 0$$

or forces acting to the left = forces acting to the right.

$$60 = 64 + 8a + 12b \qquad (1)$$

$$\Sigma V = 0$$

or forces acting upward = forces acting downward.

$$C + 80 + 15a = 5b \qquad (2)$$

$$\Sigma M = 0$$

or clockwise moments = counter-clockwise moments about any point. Let us choose the point F. The arm of the horizontal forces is $p = 2$ in. and the arm of the vertical forces is $q = 3$ in.

$$\Sigma M$$

	↷		↶	
Then	$2 \times 64 = 128$		$2 \times 60 = 120$	
	$2 \times 8a = 16a$		$3 \times C = 3C$	
	$2 \times 12b = 24b$		$3 \times 80 = 240$	
	$3 \times 5b = 15b$		$3 \times 15a = 45a$	
	$128 + 16a + 39b =$		$360 + 3C + 45a$	(3)

Equations (1), (2) and (3) may be reduced to

$$8a + 12b = -4 \qquad (1a)$$

$$15a - 5b + C = -80 \qquad (2a)$$

$$29a - 39b + 3C = -232 \qquad (3a)$$

Multiply ($2a$) by 3 and subtract ($3a$) therefrom

$$45a - 15b + 3C = -240$$

$$29a - 39b + 3C = -232$$

$$16a + 24b = -8$$

or

$$8a + 12b = -4 \qquad (4)$$

Equations ($1a$) and (4) are identical and a single solution is impossible. Equations (1), (2) and (3) represent but two independent equations and are insufficient for the solution of three unknown quantities.

(*b*) *Graphic Method.*—The magnitudes of A, B and C are determined by closing the magnitude-direction diagram. Choosing the convenient scale-ratio of 40 lb. to 1 in., we lay off oa and ab (Fig. 4*c*) representing respectively the forces E and D in magnitude, direction and sense. The diagram is to be closed from b to o by three lines which are drawn parallel respectively to the directions of A, B and C.

Through the point b draw a line parallel to the direction of either A, B or C (let us say C); and through the point o draw a line parallel to the direction of one of the two remaining forces (B, for example). The diagram may be closed by drawing any line cd parallel to the direction of the remaining force A; and we conclude, as in the algebraic method, that a single solution is impossible.

13. General Case.—By giving opposite signs to magnitudes of opposite sense, the H- and V- equations may be expressed as follows:

$$\Sigma H = 8a + 12b + 64 - 60$$
$$\Sigma V = C + 80 + 15a - 5b$$

By indicating clockwise moments as positive and counter-clockwise moments as negative, the algebraic sum of the moments of the forces about any point F is

$$\begin{aligned}\Sigma M_F &= p(64 + 8a + 12b) + 5qb - q(C + 80 + 15a) - 60p \\ &= p(8a + 12b + 64 - 60) - q(C + 80 + 15a - 5b) \\ &= (p \times \Sigma H) - (q \times \Sigma V)\end{aligned}$$

If $\Sigma H = 0$

and $\Sigma V = 0$

then $$(p \times \Sigma H) - (q \times \Sigma V) = 0$$

or $$\Sigma M_F = 0$$

for any values of p and q. Hence, if the horizontal magnitudes are balanced and the vertical magnitudes are balanced, the moments of the magnitudes are also balanced about any point chosen as the center of moments; and any equation of the M-type will be dependent upon the H- and V-equations.

Suppose that $\Sigma V = 0$ (5)

and $\Sigma M_p = (p \times \Sigma H) - (q \times \Sigma V) = 0$ (6)

it follows that $p \times \Sigma H = 0$

If $p = 0$

then Eqs. (5) and (6) are identical, and ΣH may or may not equal zero. But if p does not equal zero, as when the center of moments is not in line with the horizontal forces, then

$$\Sigma H = 0 \tag{7}$$

and Eq. (7) is dependent upon (5) and (6).

Similarly Eq. (5) is dependent upon (6) and (7) when the center of moments is not in line with the vertical forces.

Again if $\Sigma M = 0$ about two points F and G (Fig. 4*b*) then

$$\Sigma M_F = 0$$

and $$\Sigma M_G = 0$$

or $$p \times \Sigma H = q \times \Sigma V \tag{8}$$

and $$s \times \Sigma H = t \times \Sigma V \tag{9}$$

If the two points F and G are in a straight line with the point of concurrency, then

$$\frac{p}{s} = \frac{q}{t}$$

and Eqs. (8) and (9) are identical and not independent. If the two points F and G are not in a straight line with the point of concurrency, then

$$\frac{p}{s} \text{ does not equal } \frac{q}{t}$$

or $$pt - qs \text{ does not equal zero}$$

in which case, Eqs. (8) and (9) are independent.

Multiply Eq. (8) by t, and Eq. (9) by q

$$pt \times \Sigma H = qt\Sigma V$$

$$qs \times \Sigma H = qt\Sigma V$$

$$(pt - qs)\Sigma H = 0 \tag{10}$$

Since $pt - qs$ does not equal zero, then ΣH in Eq. (10) must equal zero. But if

$$\Sigma H = 0$$

then from (8) or (9)

$$\Sigma V = 0$$

Hence two equations of the type $\Sigma M = 0$ are independent if the two points representing the centers of moments are not in the same straight line as the point of concurrency; and any additional equation of either the H-, V- or M-type will be dependent upon them.

14. Only two independent statements or equations can be written for any concurrent system of coplanar forces in equilibrium, and it follows that if a single solution is possible the unknown elements in the system cannot be more than two.

15. Illustrative Problem.—Assuming that the known elements in Fig. 4*a* are the same as before, let us suppose that the magnitude of C is 10 lb., and that the sense is upward.

(*a*) *Algebraic Method.*—Equate the horizontal components of opposite sense.

$$60 = 64 + 8a + 12b$$

Equate the vertical components of opposite sense.

$$10 + 80 + 15a = 5b$$

Solve for a and b.

$$a = -5 \text{ lb.}$$
$$b = 3 \text{ lb.}$$

Therefore
$$A_h = 8a = -40 \text{ lb.},$$
$$A_v = 15a = -75 \text{ lb.},$$
and
$$A = 17a = -85 \text{ lb.}$$

The negative signs indicate that our assumption regarding the sense of the force A and its components was wrong; *i.e.*, the force acts toward the point O instead of away from it.

Likewise
$$B_h = 12b = 36 \text{ lb.},$$
$$B_v = 5b = 15 \text{ lb.},$$
and
$$B = 13b = 39 \text{ lb.}$$

The signs are positive and our assumption as to the sense of B was correct. The numerical values of the components should now be indicated in a sketch similar to Fig. 4*b*, and an arithtic check made to certify that the components balance.

(*b*) *Graphic Method.*—Returning to the point b (Fig. 4*c*) we lay off $bc' = 10$ lb. to represent the force C; draw $c'd'$ parallel to the direction of A, and $d'o$ parallel to the direction of B. Or, we may draw $c'e$ parallel to the direction of B and eo parallel

to the direction of A. The magnitude and sense of each force are thus definitely determined; in the first process by the lines $c'd'$ and $d'o$; in the second, by the lines $c'e$ and eo. By scaling either set of lines we find that the magnitudes of A and B respectively are 85 lb., acting toward the point O; and 39 lb., acting away from the point O.

16. Illustrative Problem.—Now let us suppose that all the elements of the forces in the system which we have been discussing are known except the directions (including sense) of the forces A and C, which are to be determined.

(*a*) *Algebraic Method.*—We shall assume that the direction and sense of each force A and C (Fig. 4*d*) are toward the right and upward; and let

c represent the horizontal component of C,
d represent the vertical component of C,
e represent the horizontal component of A,
and f represent the vertical component of A.

Then $$60 = 64 + 36 + c + e$$
and $$80 + d + f = 15$$

The magnitudes of A and C are 85 lb. and 10 lb. respectively, and their components are rectangular; hence

$$c^2 + d^2 = 10^2$$

and

$$e^2 + f^2 = 85^2$$

Solving the four equations for c, d, e and f, we find

$$C_h = c = \quad 0 \text{ or } \quad 8.927$$
$$C_v = d = \quad 10 \text{ or } \quad 4.507$$
$$A_h = e = -40 \text{ or } -48.927$$
and $$A_v = f = -75 \text{ or } -69.507$$

Each component has two values. The first set of values was anticipated from our previous experience with the problem; but an arithmetic check will certify that the second set is equally true. The direction and sense of each force are given by the ratio and signs of its components in each set.

(*b*) *Graphic Method.*—Lay off in Fig. 4*e* the magnitudes and directions of B, E and D, from a to d. About a and d as centers,

draw circumferences having radii equal respectively to the magnitudes of A and C, intersecting at e and f. The slopes of the lines de and df represent the directions of the force C; similarly, the lines ea and fa represent the directions of the force A.

Whenever an unknown element is the direction of a force having a known magnitude, the algebraic method involves the solution of a quadratic equation which may have two real roots—as in the present case—one real root or two imaginary roots. Graphically, the known magnitude will be represented by the radius of a circle, its circumference having two points, one point or no point in common with either another circumference or a straight line.

17. Special Case.—The system of forces represented in Fig. 5 represents a special case. The magnitudes of three forces,

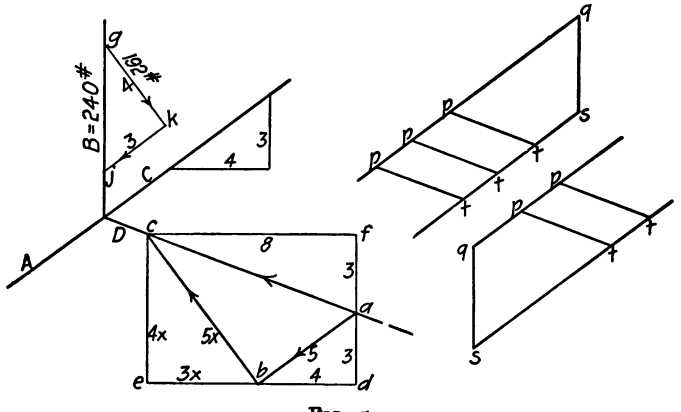


FIG. 5.

A, C and D are unknown. Since A and C have the same direction, the magnitude of D for equilibrium may be determined algebraically by equating the components normal to the direction of A and C. The normal component of the force B is

$$gk = \frac{4}{5} \times 240 = 192$$

The force D has a slope of 3 vertical to 8 horizontal, and may be resolved into the components ab and bc parallel and normal to

the line AC by the triangle abc. The triangles abd, bec and jkg are similar.

Let $ad = 3$

then $db = 4$

and $ab = 5$

Let $be = 3x$

then $ec = 4x$

and $bc = 5x$

But $4x - 3 : 3x + 4 :: 3 : 8$

therefore $bc = 5x = 7.83$

and $ac = \sqrt{5^2 + 7.83^2} = 9.29$

equating the normal components,

$$bc = gk = 192$$

Since $ac : bc :: 9.29 : 7.83$

the magnitude of the force D is

$$ac = \frac{9.29}{7.83} \times 192 = 227.8$$

In the graphic solution, we assume any magnitude and either sense for the force A as represented by pq, and lay off $qs = 240$: draw st parallel to the direction of C, and tp parallel to the direction of D. Then any line tp represents the magnitude and sense of the force D. The magnitudes of A and C cannot be determined, but their difference is represented by the difference in the lengths of pq and ts.

Hence, if a concurrent system of forces in equilibrium has three unknown magnitudes, two of which have the same direction, the magnitude of the third may be determined; but the two magnitudes having the same direction cannot be determined.

18. Conclusion.—We may conclude that in a concurrent system of coplanar forces in equilibrium, the unknown elements cannot be determined if they are more than two in number. The element of location is known for each force; hence each unknown element will be either a magnitude or a direction, and will occur in one of the four following combinations:

1. Magnitude and direction of one force.
2. Magnitudes of two forces.
3. Directions of two forces.

4. Magnitude of one force and direction of another force.

Case (2) cannot be solved if the two unknown magnitudes have the same direction. Cases (3) and (4) may have two solutions, one solution, or no solution.

19. Problems.—Solve the following problems by graphic and algebraic methods.

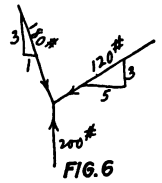


FIG. 6

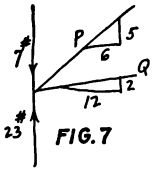


FIG. 7

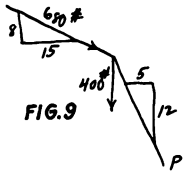


FIG. 9

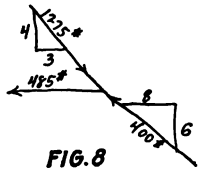


FIG. 8

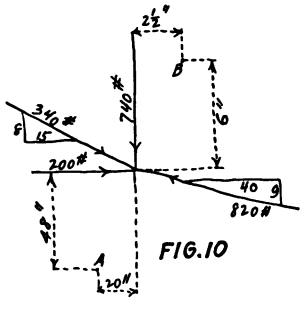


FIG. 10

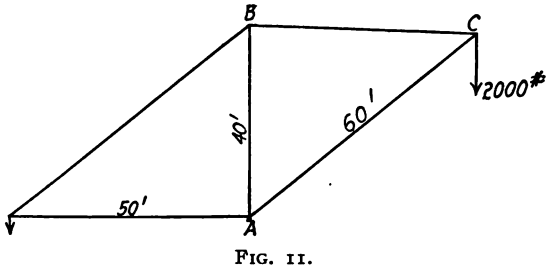


FIG. 11.

1. In Fig. 6 four concurrent forces are in equilibrium. Find the magnitude and direction of the fourth force.

2. Find the magnitudes of the forces P and Q (Fig. 7) for equilibrium.

3. Five concurrent forces are in equilibrium (Fig. 8). The magnitudes of two of the forces P and Q are 520 lb. and 340 lb. respectively. What are the directions of P and Q?

4. Four concurrent forces are in equilibrium (Fig. 9). The magnitude of Q is 300 lb. Find the magnitude of P and the direction of Q.

5. In Fig. 10 the moments of all the forces are balanced about the two points A and B, *i.e.*,

$$\Sigma M_A = 0$$

and

$$\Sigma M_B = 0$$

yet the system is not in equilibrium because neither the horizontal nor the vertical magnitudes are balanced. Explain.

6. In Fig. 11, AB represents the mast of a derrick 40 ft. long. The boom AC is 60 ft. long. The length of the cable BC can be varied to place the boom in any desired angle. The load at C is 2,000 lb. Three forces at C are in equilibrium. What is the force acting along the direction AC?

SEC. V. COPLANAR PARALLEL FORCES

20. General Considerations.—In a concurrent system the location of each force is known and we are interested in the elements of magnitude and direction only; in a parallel system

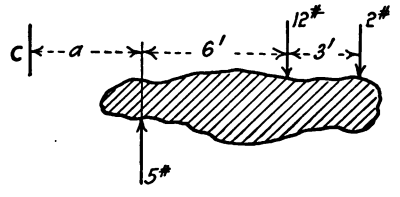


FIG. 12a.

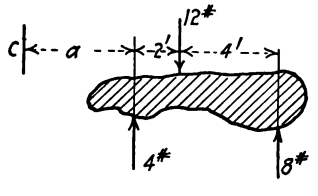


FIG. 12b.

the direction of each force is known and we have the elements of magnitude and location for our consideration.

If a body is subject to the action of a system of parallel forces which have vertical directions, the condition

$$\Sigma H = 0$$

is satisfied whether the body is in equilibrium or not. Consequently this equation lends no aid in finding the unknown elements for the equilibrium of such a system. We have the equation

$$\Sigma V = 0$$

and equations of the type

$$\Sigma M = 0$$

for the algebraic solution of a system of vertical forces.

Let us examine the expressions for ΣV and ΣM_C in connection with Figs. 12*a* and 12*b* and note the difference.

The algebraic sum of the magnitudes of the three forces in Fig. 12*a* (indicating magnitudes of upward sense as positive) is

$$\Sigma V = +5 - 12 - 2 = -9 \text{ lb.},$$

i.e., the resultant of the three forces has a magnitude of 9 lb. acting downward. The algebraic sum of the moments of the three forces about the point C (indicating clockwise moments as positive) is

$$\begin{aligned} \Sigma M_C &= -5a + 12(a + 6) + 2(a + 9) \\ &= (-5 + 12 + 2)a + 90 \\ &= -a(\Sigma V) + 90 \end{aligned}$$

Since ΣV does not equal zero, the numerical value of ΣM_C, varying with the distance a, is different for any two points not in a line parallel with the three forces. For example, when C is 6 ft. to the left of the 5-lb. force

$$a = +6$$

and $$\Sigma M_C = -6(-9) + 90 = 144 \text{ ft.-lb.}$$

When C is 3 ft. to the right of the 2-lb. force

$$a = -12$$

and $$\Sigma M_C = 12(-9) + 90 = -18 \text{ ft.-lb.}$$

The algebraic sum of the magnitudes of the three forces (Fig. 12*b*) is

$$\Sigma V = 4 - 12 + 8 = 0$$

The algebraic sum of the moments of the three forces about the point C is

$$\begin{aligned} \Sigma M_C &= -4a + 12(a + 2) - 8(a + 6) \\ &= -(+4 - 12 + 8)a - 24 \\ &= -a(\Sigma V) - 24 \end{aligned}$$

but $$\Sigma V = 0$$

therefore $$\Sigma M_C = -24$$

has the same numerical value for any distance a.

Hence, in a system of parallel forces, if

$$\Sigma M = 0$$

about some point chosen as the center of moments, it does not follow that

$$\Sigma M = 0$$

about every other point, as illustrated by Fig. 12*a*.

But when $\Sigma V = 0$

as in Fig. 12*b*, then ΣM has the same numerical value for all points chosen for the center of moments. Hence in any case where

$$\Sigma V = 0$$

if $$\Sigma M = 0$$

for any one point, it follows that

$$\Sigma M = 0$$

for all points and equilibrium is assured. There can be but one

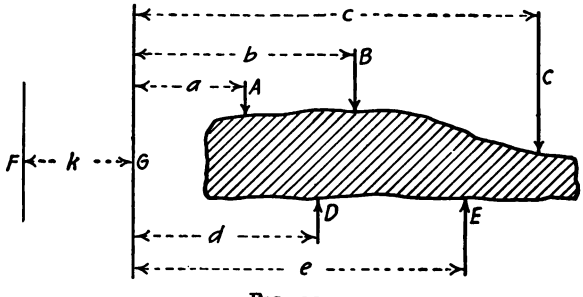


FIG. 13.

independent moment equation when

$$\Sigma V = 0$$

Let us suppose that the magnitudes and locations of the five vertical forces, acting upon the body illustrated in Fig. 13, are such that

$$\Sigma M_F = 0$$

and $$\Sigma M_G = 0$$

when F and G are any two points not in a line parallel to the system. The two equations signify that the moments of the five forces are balanced about F and G, or

$$(k + a)A + (k + b)B + (k + c)C = (k + d)D + (k + e)E \quad (1)$$

and $$aA + bB + cC = dD + eE \quad (2)$$

Subtract $$kA + kB + kC = kD + kE$$

whence $$A + B + C = D + E \quad (3)$$

or $$\Sigma V = 0$$

ence, if the moments of the forces in a parallel system balance ɔout any two points not in a line parallel to the system, the agnitudes will also balance and equilibrium is assured.

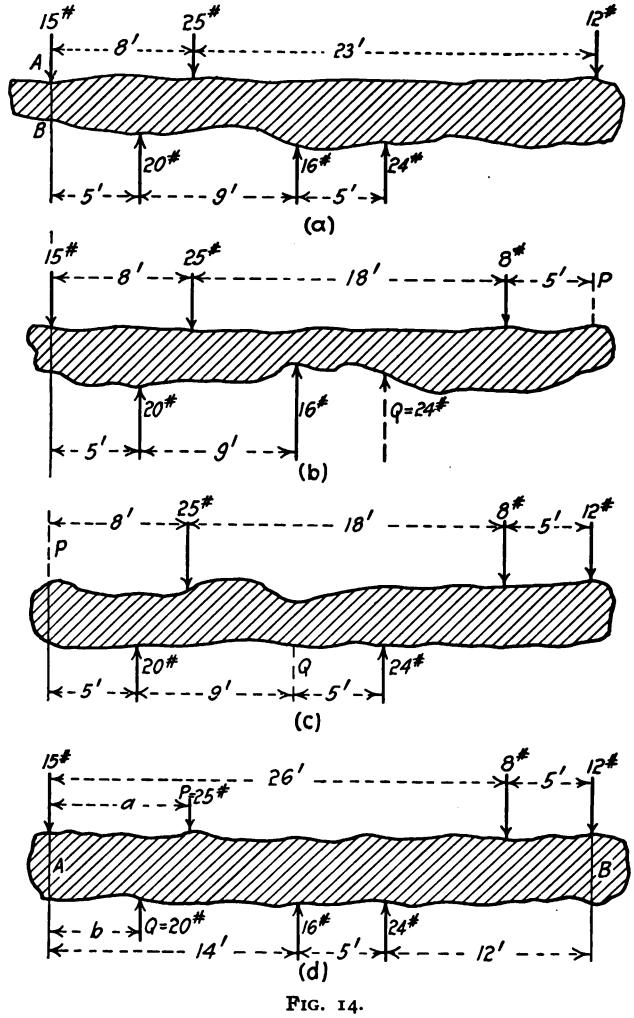


FIG. 14.

Only two of the Eqs. (1), (2) and (3) are independent, since ach may be derived from the other two.

21. Illustrative Problems.—(1) Seven vertical forces hold ıe body (Fig. 14*a*) in equilibrium. The magnitudes and

locations of six forces are known: the magnitude and location of the seventh force P is desired.

ΣV

↑	↓
20	15
16	25
24	12
60	52
52	
8	

ΣM about the line AB

↷	↶
$15 \times 0 = 0$	$20 \times 5 = 100$
$25 \times 8 = 200$	$16 \times 14 = 224$
$12 \times 31 = 372$	$24 \times 19 = 456$
572	780
	572
	$8)\overline{208}$
	26

The sum of the known magnitudes acting upward is 60 lb., and the sum of the known magnitudes acting downward is 52 lb. In order to balance the magnitudes, P must act downward having a magnitude of 8 lb. Since any point may be chosen as the center of moments, it is expedient to select a point in the line of one of the forces, thereby eliminating that force from the moment calculations. The sum of the counter-clockwise moments about the line AB is 208 ft.-lb. greater than the sum of the clockwise moments; consequently the body will rotate counter-clockwise unless the magnitude 8 lb. acting downward is located 26 ft. to the right of the line AB in order to balance the moments.

2. The location of Q (Fig. 14*b*) may be found by balancing the moments about P. The magnitude of P may be determined by balancing the magnitudes; or by balancing the moments about any point not in the line of P, after the location of Q has been found.

3. In Fig. 14*c* two magnitudes are unknown. The magnitude of Q may be found by balancing the moments about any point in the line of the force P. The magnitude of P may be found by balancing the moments about any point in the line of the force Q, or by balancing the magnitudes after the magnitude of Q has been determined.

4. In Fig. 14*d* the locations of two forces P and Q are unknown. Let a and b represent the distances from the line A of the forces P and Q respectively. The magnitudes balance, or

$$\Sigma V = 0$$

regardless of the values of a and b. Let us attempt a solution by writing two moment equations involving a and b—the one about the line A and the other about the line B.

$$\Sigma M_A$$

$$\begin{array}{rcl|rcl}
25 \times a = & & 25a & 20 \times b = & & 20b \\
8 \times 26 = & 208 & & 16 \times 14 = & 224 & \\
12 \times 31 = & 372 & & 24 \times 19 = & 456 & \\
\hline
 & 580 + 25a & = & & 680 + 20b &
\end{array}$$

or $$25a - 20b = 100$$

$$\Sigma M_B$$

$$\begin{array}{rcl|rcl}
24 \times 12 = & 288 & & 8 \times 5 = & 40 & \\
16 \times 17 = & 272 & & 25(31 - a) = & 775 & - 25a \\
20(31 - b) = & 620 & - 20b & 15 \times 31 = & 465 & \\
\hline
 & 1{,}180 & - 20b = & & 1{,}280 & - 25a
\end{array}$$

or $$25a - 20b = 100$$

The two moment equations are identical, and a single solution for the locations of the forces P and Q is impossible. This conclusion might have been anticipated. If each of the two unknown elements is a location, then all magnitudes are known and must balance for equilibrium. Since there can be but one independent moment equation when

$$\Sigma V = 0;$$

it follows that the unknown elements cannot be determined if there is more than one unknown location.

22. Conclusion.—We may conclude that in a parallel system of coplanar forces acting upon a rigid body in equilibrium, the unknown elements cannot be determined if they are more than two in number or represent more than one location. The element of direction is known for each force; hence each unknown element will be either a magnitude or a location, and will occur in one of the three following combinations:

1. Magnitude and location of one force.
2. Magnitude of one force and location of another force.
3. Magnitude of two forces.

For concurrent systems the graphic solution is generally shorter than the algebraic. There is very little to recommend the use of the graphic method in the solution of a parallel system. The algebraic method invariably gives the shorter solution. We shall refer to the graphic method in connection with parallel forces in Section VI.

SEC. VI. COPLANAR NON-CONCURRENT NON-PARALLEL FORCES

23. Transformed System.—A coplanar non-concurrent non-parallel system may contain horizontal forces, vertical forces

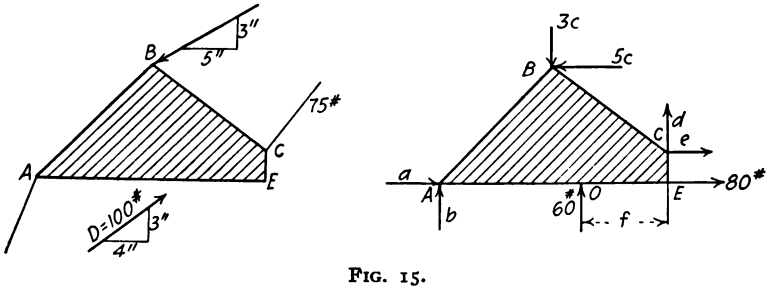


FIG. 15.

and inclined forces; and any unknown element in the system may be a magnitude, a direction or a location. Such a system may be so transformed as to contain only horizontal and vertical forces, by resolving each inclined force into its horizontal and vertical components. This transformation will change the character but not the number of the unknown elements, and is illustrated in Fig. 15.

The force located at A, having an unknown magnitude and direction, may be replaced by the two unknown magnitudes a and b having known directions and assumed senses. When the magnitudes a and b with proper senses have been determined, the magnitude, direction and sense of the original force may be found.

The force at B, having an unknown magnitude, may be replaced by the two components $3c$ and $5c$ having only one unknown quantity in the expressions which represent their magnitudes.

The magnitude of the force at C is 75 lb., but its direction is unknown. Let d and e represent the magnitudes of the two components with senses assumed. In this instance one unknown direction has been replaced by two unknown magnitudes; but one of the magnitudes may be expressed in terms of the other, since

$$d^2 + e^2 = 75^2$$

The magnitude and direction of the force D are known, but the location is unknown. It is desired to locate the force with reference to the point E. Resolve the force into its horizontal and vertical components. Their magnitudes are 80 lb. and 60 lb. respectively. Locate arbitrarily one of the components. For example, assume that the horizontal component acts through the point E. Let f represent the distance of the vertical component from E. The intersection of the two components at O marks a location of the force D. The point O will be found on the left or the right of the point E, as the solution gives a positive or a negative numerical value for f. Many different points O may be located by assigning different locations for the horizontal component; but all such points thus found will lie in the same straight line, and serve equally well in locating the force D. The slope of this line will represent the direction of the given force.

24. Illustrative Problem.—Let Fig. 16 represent a transformed system of coplanar non-concurrent non-parallel forces, after each inclined force has been replaced by its horizontal and vertical components. We wish to ascertain how many

independent equations may be written concerning the equilibrium of the rigid body upon which the forces are acting. For the solution of the four unknown magnitudes A, B, C and D, we shall attempt to write four independent equations, *i.e.* the H- and V-equations and two M-equations, one about the point O and the other about the point P.

$$\Sigma H = A + B - 25 = 0 \tag{1}$$

$$\Sigma V = C - 10 - 4 - D = 0 \tag{2}$$

$$\Sigma M_O = 10C + 12A + 40 - 400 + 10D = 0 \tag{3}$$

$$\Sigma M_P = -16B - 100 + 20C - 4A = 0 \tag{4}$$

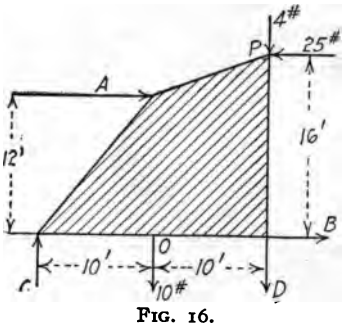


FIG. 16.

Eliminate B and C from (3) and (4) and each equation reduces to

$$12A + 20D = 220$$

This is but another way of saying that (4) adds nothing which has not been previously stated by (1), (2) and (3).

Write an M-equation for some other point and see if an identical equation cannot be derived from Eqs. (1), (2) and (3).

25. General Case.—Now consider the general case illustrated in Fig. 17.

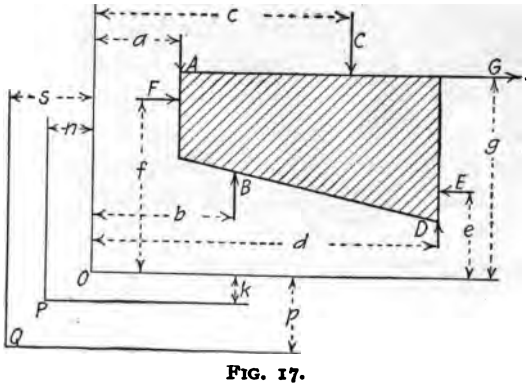


FIG. 17.

$$\Sigma V = -A + B - C + D$$
$$\Sigma H = +F + G - E$$
$$\Sigma M_P = -(a+h)A + (b+h)B - (c+h)C + (d+h)D - (f+k)F - (g+k)G + (e+k)E$$
$$\Sigma M_O = -aA + bB - cC + dD - fF - gG + eE$$
$$\Sigma M_P - \Sigma M_O = h(-A + B - C + D) + k(-F - G + E)$$
$$= (h \times \Sigma V) - (k \times \Sigma H)$$

If $\Sigma H = 0$ (1)

and $\Sigma V = 0$ (2)

then $\Sigma M_P - \Sigma M_O = 0$

or $\Sigma M_P = \Sigma M_O$

for any value of h or k.

Hence, if the vertical magnitudes are balanced and the horizontal magnitudes are balanced, the algebraic sum of the moments of the forces is the same for all points chosen as the center of moments; in which case if

$$\Sigma M = 0 \quad (3)$$

for any point, the moments are balanced about all points, equilibrium is assured; and there will be one and only one independent M-equation as illustrated in Article **24.**

26. Four Groups.—The three statements or Eqs. (1), (2) and (3) are the necessary and sufficient conditions for insuring the equilibrium of a body, when acted upon by any system of coplanar non-concurrent forces; and any fourth equation will be dependent on, and may be derived from them. Situations may arise where it will be convenient to express the conditions of equilibrium in other forms. The four ways in which the three equations may be stated are as follows:

GROUP 1	GROUP 2	GROUP 3	GROUP 4
$\Sigma H = 0$	$\Sigma M_P = 0$	$\Sigma M_P = 0$	$\Sigma M_P = 0$
$\Sigma V = 0$	$\Sigma M_O = 0$	$\Sigma M_O = 0$	$\Sigma M_O = 0$
$\Sigma M = 0$	$\Sigma H = 0$	$\Sigma V = 0$	$\Sigma M_Q = 0$

in which O, P and Q represent different points in the plane of the forces. The fact that certain restrictions must be placed upon the relative positions of these points in the last three groups, in

order that the three equations may be independent, is illustrated by the following examples.

Example 1.—Find the magnitudes A and B and the distance a (Fig. 18) for equilibrium, using the equations of group 2, and choosing any two points P and O in a vertical line.

$$\Sigma M_P = -12 + 63 - 6A - 10B - 5a = 0 \quad (1)$$

$$\Sigma M_O = 63 - 48 - 10B - 5a + 24 = 0 \quad (2)$$

$$\Sigma H = A + 6 - 8 = 0 \quad (3)$$

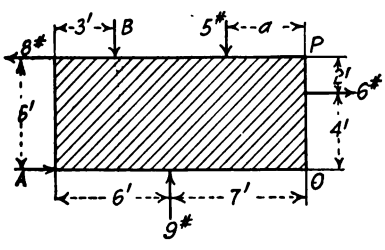


FIG. 18.

The three equations are not independent, since the third may be derived by subtracting the second from the first.

Example 2.—Use the equations of group 2 and see if a solution is possible by choosing any other two points, O and P:

(*a*) In a vertical line.

(*b*) Not in a vertical line.

Example 3.—Use the equations of group 3 and see if a solution

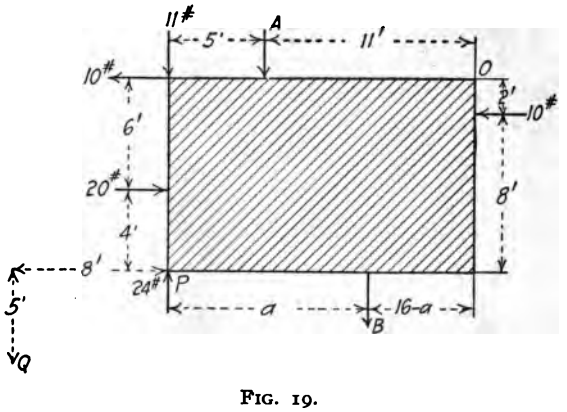


FIG. 19.

is possible by choosing any two points, O and P:

(*a*) In a horizontal line.

(*b*) Not in a horizontal line.

Example 4.—Find the magnitudes A and B and the distance a

(Fig. 19) for equilibrium using the equations of group 4 and choosing three points, O, P and Q in a straight line.

$$\Sigma M_P = 80 - 100 + 5A - 80 + aB = 0 \quad (1)$$
$$\Sigma M_O = 20 - (16 - a)B + 384 - 120 - 176 - 11A = 0 \quad (2)$$
$$\Sigma M_Q = 180 - 150 + 88 + 13A - 130 + (8 + a)B - 192 = 0 \quad (3)$$

Multiply Eq. (3) by 2, add Eq. (2), and divide the sum by 3. The quotient is Eq. (1). Only two independent equations are represented, since any one of the three may be derived from the other two, and a solution of the problem as stated is impossible.

Example 5.—Choose three points O, P and Q *not* in a straight line and see if a solution is possible.

It has been shown that *under certain conditions* the three equations in each of the last three groups represent but two independent equations, and are insufficient for the solution of three unknown quantities. The general cases will now be considered.

27. *Group* 2.—Suppose that in Fig. 17

$$\Sigma M_P = 0 \quad (1)$$
$$\Sigma M_O = 0 \quad (2)$$
and $$\Sigma H = 0 \quad (3)$$
then $$\Sigma M_P - \Sigma M_O = (h \times \Sigma V) - (k \times \Sigma H) = 0 \quad (4)$$

If the two points P and O are in a vertical line

$$h = 0$$
hence $$h \times \Sigma V = 0$$

for any value of ΣV, in which case,

(*a*) from Eq. (4)

$$k \times \Sigma H = 0$$

and, since k is not zero, $\Sigma H = 0$ and it is obvious that Eq. (3) may be derived from Eq. (4) which was derived from Eqs. (1) and (2) and is dependent upon them. Hence, when the two points P and O are in a vertical line, Eqs. (1), (2) and (3) represent but two independent equations (see Examples 1 and 2*a*).

(*b*) the value of ΣV may or may not equal zero and equilibrium is not assured.

But if the two points P and O are not in a vertical line, h in Eq. (4) does not equal zero, in which case

(*c*) $$k \times \Sigma H$$

in Eq. (4) may or may not equal zero; hence ΣH may or may not equal zero and Eq. (3) is independent of Eqs. (1) and (2) (see Example 2*b*).

(*d*) since from Eq. (3)

$$\Sigma H = 0$$

then from Eq. (4)

$h \times \Sigma V = 0$ and since h does not equal zero

then $$\Sigma V = 0 \qquad (5)$$

and equilibrium is assured. Equation (5) does not represent an independent equation since it is derived from (1), (2) and (3).

28. Group 3.—Similarly it may be demonstrated that Eq. (3) is dependent upon Eqs. (1), (2) and (5) when the two points P and O are not in a horizontal line (see Examples 3*a* and 3*b*).

29. Hence in any case where the moments of all the forces in a non-concurrent non-parallel system are balanced about any two points in a diagonal line, if the horizontal magnitudes are balanced the vertical magnitudes are also balanced; or if the vertical magnitudes are balanced, the horizontal magnitudes are also balanced. Only the *H- or* the *V*-equation can be independent of the two moment equations.

30. Group 4.—Suppose that in Fig. 17

$$\Sigma M_P = 0 \qquad (1)$$

$$\Sigma M_O = 0 \qquad (2)$$

and $$\Sigma M_Q = 0 \qquad (6)$$

then $$\Sigma M_P - \Sigma M_O = (h \times \Sigma V) - (k \times \Sigma H) = 0 \qquad (7)$$

and $$\Sigma M_Q - \Sigma M_O = (s \times \Sigma V) - (p \times \Sigma H) = 0 \qquad (8)$$

If the three points O, P and Q are in a straight line

$$\frac{h}{s} = \frac{k}{p}$$

and Eqs. (7) and (8) are identical and not independent (see Example 4). Eliminate ΣV in (7) and (8)

$$\frac{k \times \Sigma H}{h} = \frac{p \times \Sigma H}{s}$$

or
$$\frac{h \times \Sigma H}{s} = \frac{k \times \Sigma H}{p} \tag{9}$$

If the three points O, P and Q are *not* in a straight line, then

$$\frac{h}{s} \text{ does not equal } \frac{k}{p}$$

in which case Eqs. (7) and (8) are independent; and ΣH in Eq. (9)must equal zero (see Example 5). But if

$$\Sigma H = 0$$

then from Eq. (7) or (8)

$$\Sigma V = 0$$

31. Hence in any case where the moments of all the forces in a non-concurrent non-parallel system are balanced about any three points *not* in a straight line, the horizontal magnitudes are balanced, the vertical magnitudes are balanced, and the H- and V-equations will be dependent upon the three M-equations.

32. One Unknown Location.—In Fig. 20 the unknown loca-

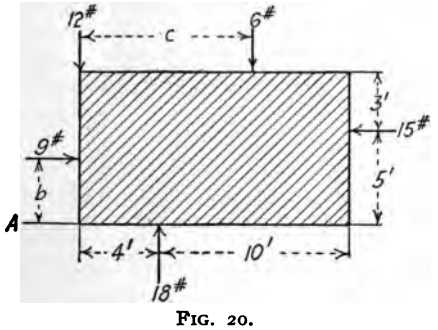


FIG. 20.

tions of two forces are represented by the quantities b and c. Let the horizontal magnitude A represent a third unknown element. All the vertical magnitudes are known and balanced. The magnitude A, determined by balancing the horizontal magnitudes, equals 6 lb., and acts to the right. Since the magnitudes are now balanced horizontally and vertically, it is possible to have only one independent M-equation for the solution of b and c, and a single solution is impossible.

33. Show that a similar condition is encountered when the unknown elements are two locations and one direction. There can be only one unknown location if a single solution is possible.

34. Combinations.—The unknown elements which can be wholly or partially determined in a coplanar non-concurrent non-parallel system of forces will appear in one of the following combinations; where P, Q and S represent any forces which have unknown elements

1. The magnitude of P, the magnitude of Q and the magnitude of S.
2. The direction of P, the direction of Q and the direction of S.
3. The magnitude of P, the magnitude of Q and the direction of S.
4. The magnitude of P, the magnitude of Q and the direction of Q.
5. The magnitude of P, the magnitude of Q and the location of S.
6. The magnitude of P, the magnitude of Q and the location of Q.
7. The direction of P, the direction of Q and the magnitude of S.
8. The direction of P, the direction of Q and the magnitude of Q.
9. The direction of P, the direction of Q and the location of S.
10. The direction of P, the direction of Q and the location of Q.
11. The magnitude of P, the direction of Q, and the location of S.
12. The magnitude of P, the direction of P and the location of P.
13. The magnitude of P, the direction of P and the location of Q.
14. The magnitude of P, the direction of Q and the location of P.
15. The magnitude of P, the direction of Q and the location of Q.

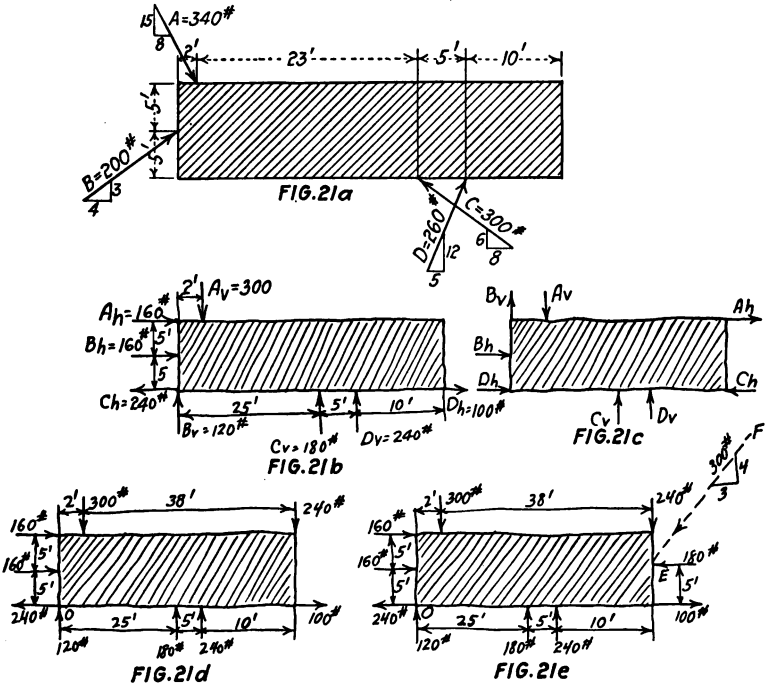
Cases (1), (4) and (12) are more frequently encountered in the theory of framed structures than any or all of the others. Several cases are indeterminate under certain conditions. Case (1) is in this class when P, Q and S have the same location, or are parallel. Other cases are partially indeterminate, for they may have several solutions or none. Any problem which involves a force having a known magnitude and an unknown direction is in this class. The most difficult problems are to be found in cases (2) and (7).

35. Illustrative Problems.—1 Determine the unknown elements necessary for equilibrium (Fig. 21*a*).

(*a*) *Algebraic*[1] *Method.*—The three elements of each of the

[1] The term "algebraic" is used simply for the purpose of distinguishing the method of computing by the use of numerals or letters, from the graphic method

four forces shown are given. The elements necessary for equilibrium are the magnitude, direction and location of a fifth



force. The problem is one of case (12). Make a sketch (Fig. 21*b*) and transfer to it the horizontal and vertical components of the four known forces properly located.

---

in which quantities are represented by the length, direction and location of lines. Some writers refer to the two methods as the analytic and graphic; but this is manifestly incorrect, since both methods are analytic. The frequent use however, of the word "algebraic" should be discouraged. It is true that algebraic equations have been used in the discussions which have preceded, but in the actual solution of problems their use is often a hindrance rather than a help. In the first place, if the equations are not written, the problem can frequently be solved more quickly and easily; and the computations can be arranged in a more convenient form for checking. But a more important reason for discarding the equation whenever possible is the fact that by so doing, the attention is held fast to the original problem in statics, and not drawn far afield by an exercise in elementary algebra.

$8^2 = 64$
$15^2 = 225$
$\sqrt{289} = 17$
$A_h = \frac{340}{17} \times 8 = 160$
$A_v = \frac{340}{17} \times 15 = 300$

The horizontal and vertical components of A are 160 lb. and 300 lb. respectively, and are so placed on the sketch as to intersect at a point in the line of the force A. The components of the other known forces are determined and located in a similar manner. (The sketch in Fig. 21*c* is superfluous, except that it shows one of the several other forms in which the components might have been correctly placed.)

ΣV

→	←
160	240
160	
100	
420	240
240	
180	

The sum of the horizontal components acting to the right is greater by 180 than the sum of the horizontal components acting to the left; consequently the body will move to the right unless a horizontal component having a magnitude of 180 acts to the left to balance the horizontal forces.

ΣH

↑	↓
120	300
180	
240	
540	300
300	
240	

A vertical component having a magnitude of 240 must act downward to balance the vertical forces. Locate one of the components arbitrarily. Let us say that the vertical component acts at the upper right-hand corner of the body as shown in Fig. 21*d*. Find the algebraic sum of the moments of the magnitudes about any point.

↷	ΣM_0	↶	
$5 \times 160 =$	800	$25 \times 180 =$	4,500
$10 \times 160 =$	1,600	$30 \times 240 =$	7,200
$2 \times 300 =$	600		
$40 \times 240 =$	9,600		
	12,600		11,700
	11,700		
	180)900		
	5		

The point O is chosen, for the moments of the three magnitudes which pass through it are thereby eliminated. The clockwise moments about the point O are greater by 900 ft.-lb. than the counter clockwisemoments. The body will rotate clockwise about the point O unless the horizontal component with its magnitude of 180, which must act to the left, passes through the point E at a distance 5 above the point O (Fig. 21*e*). Check the computations. See that the magni-

tudes are balanced horizontally and vertically, and that the moments are balanced about some point other than *O*.

$180^2 = 32{,}400$
$240^2 = 57{,}600$
$\therefore \sqrt{90{,}000} = 300$

The magnitude of the required force is 300; its sense is downward to the left; it is located by the point *E*; and its direction *FE*, determined by the ratio of its components, is 4 vertical to 3 horizontal. The Figs. 21*c*, *d* and *e* are unnecessary for the solution; but were sketched to illustrate the successive steps in the solution.

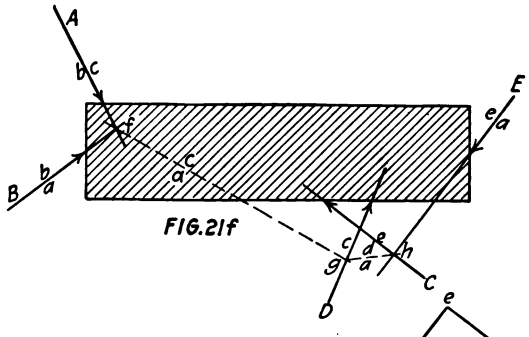
Instead of locating the vertical component at the upper right-hand corner, locate the horizontal component at some point and determine the location of the vertical component. Note that the two components thus located intersect at some point in the line *EF*.

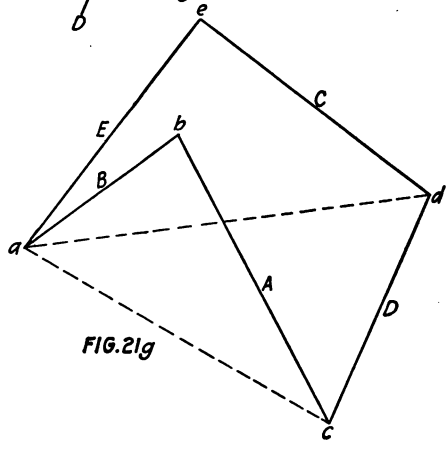
Graphic Method.—Since a concurrent system of forces cannot result in rotation about the point of concurrency, equilibrium is assured if the magnitude-direction diagram is a closed polygon and the sense is continuous. While the closed magnitude-direction diagram is a *necessary condition* for equilibrium in a non-current system, it is not a *sufficient condition*. The closed magnitude-direction diagram takes no account of the location of forces and insures against translation only. In order that there may be no rotation, another figure, which will be called the location-direction[1] diagram must also be drawn, but not necessarily closed.

[1] One of the two diagrams which are drawn for a graphic solution of a system of coplanar non-concurrent forces is usually called the "force polygon"; the other, the "equilibrium polygon." These terms do not seem to describe adequately the essential qualities of the two diagrams. Any line in the force polygon, so called, does not completely represent a force. It represents only two of the three elements, *viz.*, magnitude and direction. The term "equilibrium polygon" seems to be applicable equally to both diagrams, since both are requisite for equilibrium. Any line in the equilibrium polygon, or as it is sometimes called, the funicular polygon or string polygon, also represents but two of the three elements of a force, *viz.*, location and direction. Hereafter, we shall use the distinctive terms, magnitude-direction diagram and location-direction diagram. The lines in the first diagram will be called magnitude-directions; in the second, location-directions. The two diagrams have nothing in common except the element of direction and the sense, which are registered in both. There is nothing pertaining to locations in the first, and nothing pertaining to

The closing of the magnitude-direction diagram in the graphic method corresponds to the solution of $\Sigma H = 0$ and $\Sigma V = 0$ in the algebraic method; while the drawing of a location-direction diagram corresponds to the solution of $\Sigma M = 0$.

The body in Fig. 21*a* is re-drawn to scale (Fig. 21*f*) and full lines are added to represent known location-directions. The





lengths of these lines have no significance whatever, since they do not represent magnitudes in any particular. The magnitude-direction diagram is now constructed (Fig. 21*g*) from *a* to *e* by laying off to scale the known magnitude-directions in any order and in the same manner as for a concurrent system. The diagram is closed by drawing the magnitude-direction *ea* which

magnitudes in the second. There is no connection between the scale ratios used in laying off the elements in the two diagrams; for in the first the scale ratio is pounds-to-the-inch, in the second, feet-to-the-inch.

gives the magnitude, direction and sense of the equilibrant *E*. The location of the equilibrant with reference to the body (Fig. 21*f*) remains to be determined. This may be accomplished by drawing the location-direction diagram in one of two ways:

(*a*) By combining the forces in pairs and drawing the location-directions of their resultants.

(*b*) By resolving the forces and drawing the location-directions of their components.

The first method is the simpler and more direct way, and will now be considered.

(*a*) By Drawing the Resultants.—The magnitude-direction and the location-direction of a force are designated by the same letters, placed at the ends of the magnitude-direction and on opposite sides of the location-direction. Obviously the magnitude-direction and the location-direction of any force are alike in sense and are parallel. The magnitude-direction of the resultant of the forces *A* and *B* is *ac*; the location-direction of this resultant is *ac* drawn through *f*: it intersects the location-direction of the force *D* at *g*. The magnitude-direction of the resultant of the forces *ac* and *D* is *ad;* the location-direction of this resultant is *ad* drawn through *g:* it intersects the location-direction of the force *C* at *h*. The magnitude-direction of the resultant of the forces *ad* and *C* is *ae*; the location-direction of this resultant is *ae* drawn through *h*. The resultant and the equilibrant of a system of forces have the same magnitude, direction and location; they differ in sense only. Hence, the location-direction of the equilibrant *E* is the line *ea*, and any point in it may serve as a location.

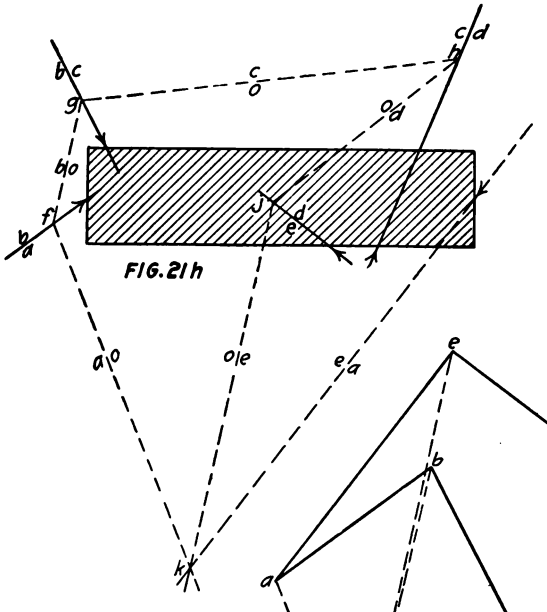
Compare the results of the graphic and algebraic solutions.

Draw the magnitude-direction and the location-direction *ce*. Explain why the location-directions *ac* and *ce* intersect on the location-direction *ae*.

The location-direction diagram is not a closed figure when constructed by drawing resultants.

(*b*) By Drawing the Components.—The method of drawing the location-directions of the resultants, while short and simple, obviously could not have been employed if the directions of the forces had been parallel; for there would have been no points

of intersection in the location-direction diagram. The method of drawing the location-directions of the components of forces has a wider range of applicability than the preceding method, but necessitates the drawing of more lines. The procedure for finding the magnitude, direction and sense of the equilibrant



is the same as in the foregoing solution; only in the method of locating the equilibrant are the two solutions different.

The full lines (Fig. 21*h*) represent known location-directions. The magnitude-direction diagram has been closed (Fig. 21*k*). Choose any convenient point *o*, called the pole; and draw the magnitude-directions *oa*, *ob*, *oc*, *od* and *oe*. The pole *o* should not be taken on a line passing through any two adjacent apexes. Select another convenient point on a known location-direction,

as *f* on *ab*, and draw the location-directions *ob*, *oc* and *od* as indicated; thus establishing the points *g*, *h* and *j*. Through *f* and *j* draw the location-directions *oa* and *oe* respectively. Their intersection at *k* closes the location-direction diagram and determines the location-direction *ea* of the equilibrant.

The three forces acting at *f* are in equilibrium, since their location-directions *ab*, *bo* and *oa* are concurrent and their magnitude-directions *ab*, *bo* and *oa* form a closed diagram. The concurrent forces acting at *g*, *h*, *j* and *k* are also in equilibrium for a similar reason. The two equal and opposite forces acting along the line *fg*; the one *bo* toward *f*, the other *ob* toward *g*, are balanced. The same is true with respect to the two forces acting along each of the other sides of the location-direction diagram.

Although an infinite number of location-direction diagrams may be drawn by choosing different points for the pole *o* or the starting point *f;* yet in each diagram thus drawn, the location-directions *ao* and *oe* will invariably intersect, not at the same point, but on the same line. This line represents the locus of the location-direction *ea*.

It has been shown that when $\Sigma H = 0$ and $\Sigma V = 0$, if $\Sigma M = 0$ about any point, then $\Sigma M = 0$ for all points. Likewise when the magnitude-direction diagram closes, it follows that if one location-direction diagram closes, all will close.

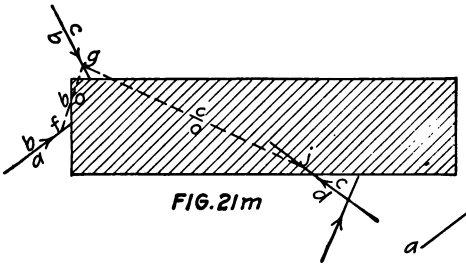
Some care must be exercised in selecting the points *o* and *f*, lest the location-direction diagram fall beyond the limits of the drawing paper.

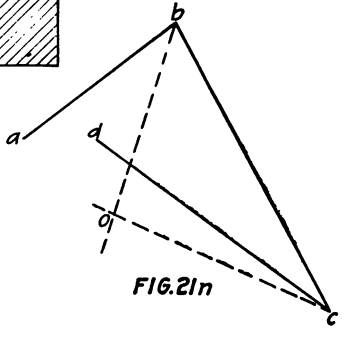
A collapsible frame composed of five members and having the configuration of the diagram *fghjk*, if substituted for the rigid body, will hold the five forces in equilibrium. The equilibrium will be unstable however, for the frame will collapse if only a slight change be made in its shape. The same is true of any one of the infinite number of frames which may be constructed by choosing different points *o* and *f* in this or in any other problem in which the frame has four or more members, and thereby made susceptible to a collapse.

Location-direction Drawn Through Given Points.—An infinite number of location-direction diagrams may be drawn through any given point *p* on a location-direction, by choosing

different positions for the pole o and beginning the location-direction diagram at p in each instance. An infinite number of location-direction diagrams may be drawn through two points, p on one location-direction and s on another. Call the location-directions on which the two points p and s are located xy and yz, and draw the location-direction oy connecting p and s. In constructing the magnitude-direction diagram, arrange the order so that the magnitude-directions xy and yz are adjacent. Draw the magnitude-direction oy and select any point o on it for the pole.

Three points f, g and j (Fig. 21*m*) may be selected at random





on the location-directions of three forces; and a location-direction diagram drawn through them, provided the points are not in a straight line. Call the location-directions on which f, g and j are situated, ab, bc and cd respectively; and draw the location-directions ob and oc through f, g and j as indicated. Begin the magnitude-direction diagram (Fig. 21*n*) so that bc will appear between ab and cd. Draw the magnitude-directions ob and oc intersecting at o, the desired pole.

Complete the magnitude-direction diagram in two ways; thus giving two location-direction diagrams through f, g and j.

Evidently the points f, g and j may be connected by location-directions in two other ways; *viz.*, by connecting f and g, and f and j; or by connecting f and j, and g and j. Thus, several

constructions of the location-direction diagram through the three given points *f*, *g* and *j* are possible; but the number for any given system is limited, and depends upon the number of forces in the system.

2. Four vertical forces *ab*, *bc*, *cd* and *de* (Fig. 22*a*) are completely known. The direction of the force *ef* is vertical. Find the magnitude of *ef* and the magnitude and direction of *fa* for equilibrium.

Graphic Method.—Lay off the magnitude-directions (Fig. 22*b*) from *a* to *e*. The magnitude-direction diagram must be closed from *e* to *a* by two magnitude-directions *ef* and *fa*. Since the

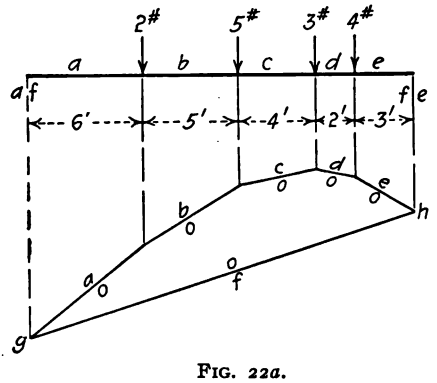


FIG. 22*a*.

FIG. 22*b*.

location-direction *ef* is vertical, the magnitude-direction *ef* is also vertical and the point *f* must fall somewhere on the line *ea*, making the magnitude-direction *fa* vertical. Draw the location-direction *fa*. Choose any pole *o* and draw the magnitude-directions *oa*, *ob*, *oc*, *od* and *oe*. Select any point *g* on the location-direction *fa*; and draw the location-directions *oa*, *ob*, *oc*, *od* and *oe*; thereby locating the point *h*. Close the diagram from *h* to *g* by drawing the location-direction *of*. Draw the magnitude-direction *of* which determines the magnitude-directions *ef* and *fa*. Scale the magnitude-directions *ef* and *fa*, and check by the algebraic method.

The magnitude-direction diagram becomes a straight line in a parallel system.

In engineering practice the rigid body upon which the forces

are acting is called a beam. The forces acting downward are known as loads and the upward forces are called reactions.

3. The body in Fig. 23*a*, weighing 1,100 lb. and supported at two points X and Y, resists a horizontal force of 350 lb. The vertical gravity line of the body is 10 ft. from the support at Y. The reaction at Y is 650 lb. Find the direction of the reaction at Y and the magnitude and direction of the reaction at X.

Algebraic Method.—Make the sketch (Fig. 23*b*) assuming H- and V-components for the reactions. Balance the moments of the forces about the left support and thus eliminate c and d;

$$12a - 5b = 5{,}950$$

Also $$a^2 + b^2 = 650^2$$

whence $$a = 600 \text{ or } 244.97$$

and $$b = 250 \text{ or } -602.07$$

Balance the H- and V-magnitudes

$$c = 600 \text{ or } -252.07$$

$$d = 500 \text{ or } 855.03$$

The two solutions should be indicated in separate sketches, and the directions of the right reaction and the magnitudes and directions of the left reaction computed therefrom.

Graphic Method.—Lay off the known location-directions AB and BC (Fig. 23*a*), and the known magnitude-directions ab and bc (Fig. 23*c*). The magnitude-direction cd has a known length but an unknown direction; therefore the point d in the magnitude-direction diagram will fall on the circumference drawn about c, with a radius representing 650 lb. When d is located, the magnitude-direction diagram may be closed. Through the intersection e of the location-directions AB and BC, draw their resultant location-direction AC. Only one point on each of the location-directions CD and DA is known; *viz.*, the point of each support. Draw the location-direction OD connecting the points X and Y, and close a location-direction diagram by drawing the location-directions OA and OC intersecting at any point f on the location-direction AC. Draw the corresponding magnitude-directions oa and oc intersecting at o. Through o draw the magnitude-directions

od and *od'* cutting the circumference at *d* and *d'*. The magnitude-direction diagram may now be closed by drawing the magnitude-directions *cd* and *da*, or by drawing *cd'* and *d'a*, thus giving two solutions.

For each point *f* chosen on the location-direction *AC*, there is a corresponding pole *o* on the line *dd'*. It was unnecessary,

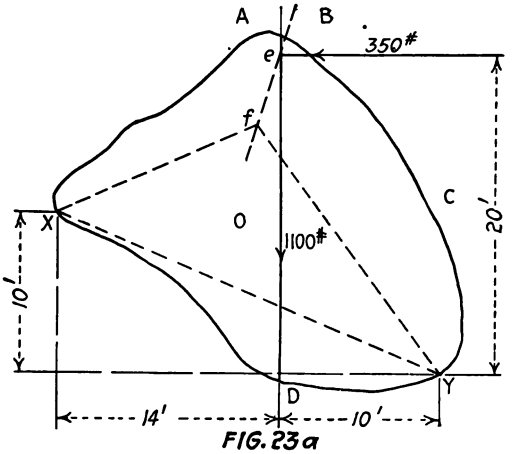


FIG. 23a

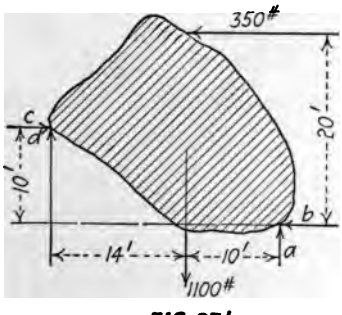


FIG. 23b

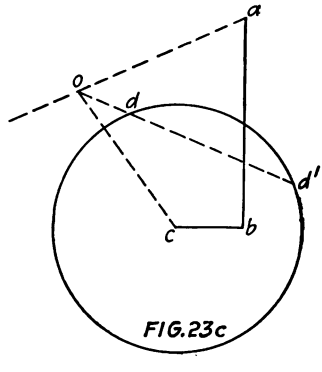


FIG. 23c

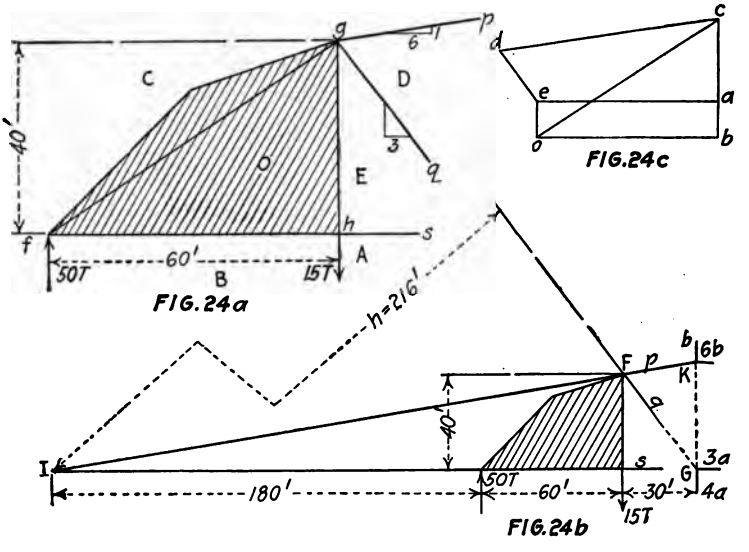
except for illustration, to draw the location-direction *AC*; for the location-direction diagram could have been closed at *e* as well as at any other point *f* on the location-direction *AC*.

Complete the magnitude-direction diagram, and compare these results with the two algebraic solutions.

4. The body in Fig. 24*a*, carrying a vertical load of 15 tons, is supported by a vertical reaction of 50 tons; and three other

forces p, q and s having known locations and directions. Find the magnitudes p, q and s if the weight of the body is neglected.

Algebraic Method.—Make the sketch (Fig. 24*b*) and assume H- and V-components for the inclined forces p and q. The unknown quantities a, b and s may be determined by writing and solving three independent static equations. There is, however, a very simple and more direct method whereby any one of the unknown magnitudes may be determined independ-



ently of the other two. The method of procedure is as follows: balance the moments of all the forces about the point F, thereby eliminating the two unknown magnitudes p and q and the load of 15 tons. Only two magnitudes remain—the reaction of 50 tons and the magnitude s—and their moments about F must balance for equilibrium; for the other magnitudes cannot help or hinder rotation about the point F. The moment of the reaction about F is 3,000 ft.-tons clockwise; and the body will rotate clockwise about F unless the magnitude s, acting through a distance of 40 ft. and to the right or away from the body, is 75 tons.

At the point G, where the directions of q and s intersect,

resolve the magnitude q into H- and V-components $3a$ and $4a$. At the point K in a vertical line with G, resolve the magnitude p into H- and V-components $6b$ and b. Balance the moments of all the forces about G, thereby eliminating the unknown magnitudes s, q and b. The H-component of p is 90 tons

$$\begin{aligned} 90 \times 50 &= 4{,}500 \\ 30 \times 15 &= \underline{\ 450} \\ 45)&\underline{4{,}050} \\ 90 &= 6b \\ b &= 15 \end{aligned}$$

acting to the left. The V-component of p is 15 tons acting downward.

$p = \sqrt{90^2 + 15^2} = 91.24$ tons acting toward the body.

The directions of p and s intersect at I, 180 ft. to the left of the reaction; and by balancing the moments of all the forces about I, the magnitudes p, s and $3a$ are eliminated.

$$\begin{aligned} 180 \times 50 &= 9{,}000 \\ 240 \times 15 &= \underline{3{,}600} \\ 270)&\underline{5{,}400} \\ 20 &= 4a \end{aligned}$$

The V-component of q is 20 tons acting downward. The H-component of q is 15 tons acting to the right.

$q = \sqrt{15^2 + 20^2} = 25$ tons acting away from the body.

The magnitude q may also be determined by dividing the moments of all the forces about I by the arm h. The distance FG is 50 ft.

$$\begin{aligned} 40 : 50 &:: h : 270 \\ h &= 216 \end{aligned}$$

$$q = \frac{5{,}400}{216} = 25 \text{ tons acting away from the body.}$$

The magnitudes $3a$ and $4a$ could have been determined by balancing the H- and V-magnitudes; after the magnitudes s, b and $6b$ had been determined. Check the solution already determined by this process.

The sketch (Fig. 24b) as shown is incomplete; since each numerical value, as soon as it is determined, should be sub-

stituted for its symbol *on the sketch* and an arrow-head added to indicate the sense.

It has been stated that the use of algebraic equations should be avoided whenever possible. The present case is a good illustration. The arithmetic solution is not only shorter; but

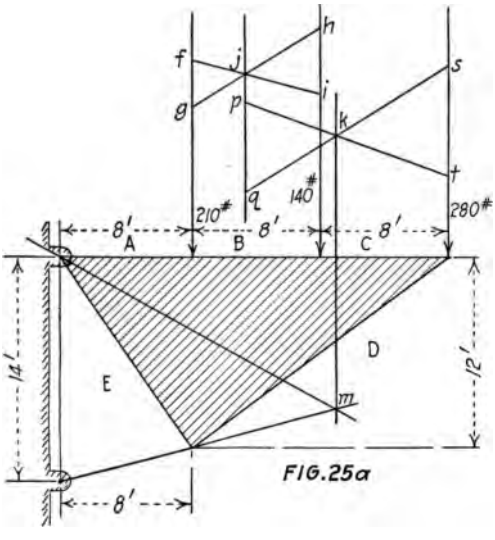


FIG. 25a

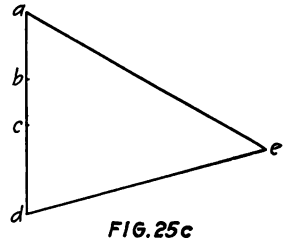


FIG. 25c

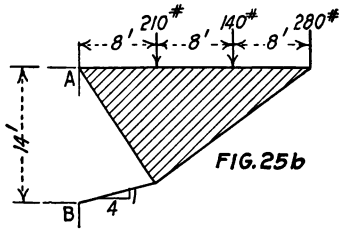


FIG. 25b

also gives the student an opportunity for the exercise of his judgment in choosing the best method of attacking the problem in order that the solution may be accurately and quickly made, and his computations advantageously arranged for checking.

Graphic Method.—The location-directions of all the forces are known. They are drawn to scale in Fig. 24*a* and indicated by the symbols *AB*, *BC*, *CD*, *DE* and *EA*. The known magnitude-directions *ab* and *bc* are laid off to scale in Fig. 24*c*. Draw the location-directions *OB*, *OC* and *OE*. The forces acting

at each of the vertices f, g and h, of the location-direction diagram thus formed, may represent three concurrent systems in equilibrium. Close the magnitude-direction diagram for the point f by drawing the magnitude-directions *co* and *ob*, intersecting at *o*. Two magnitude-directions for the point h are now known—*ab* and *bo*. Close the diagram by drawing the magnitude-directions *oe* and *ea*, intersecting at *e*. Two magnitude-directions for the point g are known—*eo* and *oc*. Close the diagram by drawing the magnitude-directions *cd* and *ed*, intersecting at *d*. The magnitude-direction diagram *abcdea* is closed and the magnitudes *cd*, *de* and *ea* are determined.

Compare the results with those obtained by algebraic solution.

5. The body in Fig. 25*a* supports three loads. What are the reactions if the weight of the body is neglected?

Algebraic Method.—Remove the body from the supports and indicate the H- and V-components of the forces necessary for equilibrium, as shown in the sketch (Fig. 25*b*). Balance the moments of all the forces about A.

$8 \times 210 = 1{,}680$
$16 \times 140 = 2{,}240$
$24 \times 280 = 6{,}720$
$14)10{,}640$
760 = H-component at B, acting to the right.
= H-component at A, acting to the left.
$\frac{760}{4} = 190$ = V-component at B acting upward.
210
140
280
630
190
440 = V-component at A acting upward.
$440^2 = 193{,}600$
$760^2 = 577{,}600$
$\sqrt{771{,}200} = 878.18$ = magnitude of the force at A.
$\frac{440}{760} = \frac{11}{19}$ The direction of the force at A has the slope of 11 vertical to 19 horizontal.
$190^2 = 36{,}100$
$760^2 = 577{,}600$
$\sqrt{613{,}700} = 783.39$ = magnitude of the force at B.

Change the location of the components of the lower reaction so that, by balancing the moments of all the forces about A, the H-component of the lower reaction is eliminated.

Graphic Method.—The location-directions of the known forces AB, BC and CD (Fig. 25*a*) do not intersect; and the conventional method of constructing the location-direction diagram by drawing the location-directions of the components of the forces, somewhat after the manner illustrated in Problem 2, would ordinarily be followed. But by a simple expedient the location-direction diagram may be more quickly drawn by locating the resultants instead of the components. Draw the magnitude-direction diagram (Fig. 25*c*) from a to d. Select two points f and g anywhere on the location-direction AB so that fg will represent the magnitude bc. Similarly let hi, on the location-direction BC, represent the magnitude ab. The intersection of the two lines fi and gh at j marks the location of the resultant ac of the two forces ab and bc. In like manner pq represents the magnitude cd; st represents the magnitude ac; and k marks the location of the resultant ad of the forces ab, bc and cd. The location-direction DE is known, and intersects the location-direction AD at m; through which point the location-direction EA must pass if there is to be no rotation of the body. The location-directions of all the forces are known. Close the magnitude-direction diagram by drawing the magnitude-directions de and ea.

Compare the results with those obtained by the algebraic solution.

If the sense of ab had been upward, show that the resultant of ab and bc would have scaled 70 in the magnitude-direction diagram; and its location would have been at the point of intersection of the two lines through hf and ig.

36. Problems.

1. In Fig. 26 the moments of all the forces are balanced about the three points O, P and Q; *i.e.*

$$\Sigma M_O = 0$$
$$\Sigma M_P = 0$$
$$\Sigma M_Q = 0$$

yet the body is not in equilibrium, for the horizontal magnitudes do not balance, neither do the vertical magnitudes balance. Explain.

2. The homogeneous body of uniform thickness in Fig. 27 weighs 50 lb. and is supported at A, B and C. The reaction at A is 25 lb. The reactions at B and C are vertical and horizontal respectively, as indicated by the roller supports. Determine the unknown elements for equilibrium.

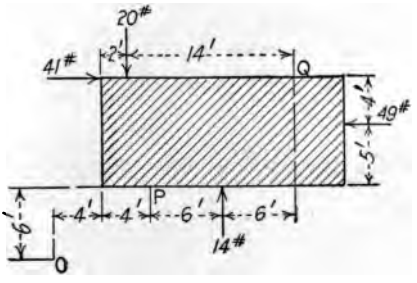


FIG. 26.

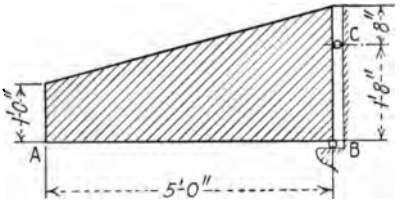


FIG. 27.

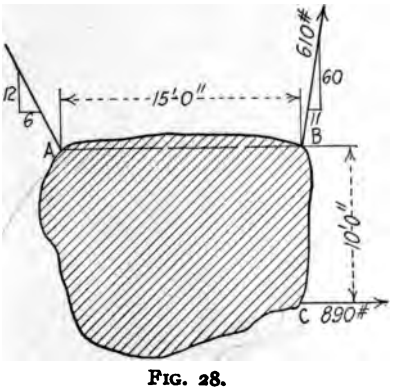


FIG. 28.

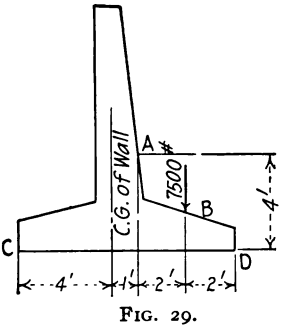


FIG. 29.

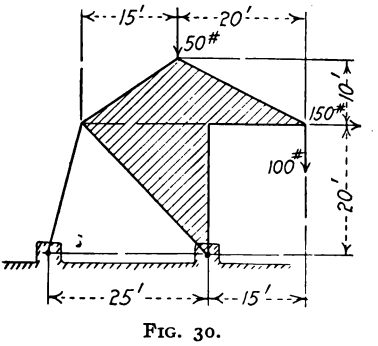


FIG. 30.

3. The body in Fig. 28 is supported at A, B and C. Find the weight of the body, the magnitude of the force at A, and locate the vertical gravity line of the body.

4. The cross-section of a retaining wall weighing 6,000 lb. per linear foot is; represented in Fig. 29. The resultant earth pressures per linear foot are 15,000 lb. at A and 7,500 lb. at B. What is the direction of the resultant pressure at A, if the resultant reaction of the ground on the base CD is 25,500 lb. per linear foot; and what is the total friction between the ground and wall on the line CD? At what point on the line CD does the resultant reaction act?

5. Neglect the weight of the body in Fig. 30 and determine the reactions.

6. The body in Fig. 31 weighs 2,000 lb. Locate the vertical gravity line;

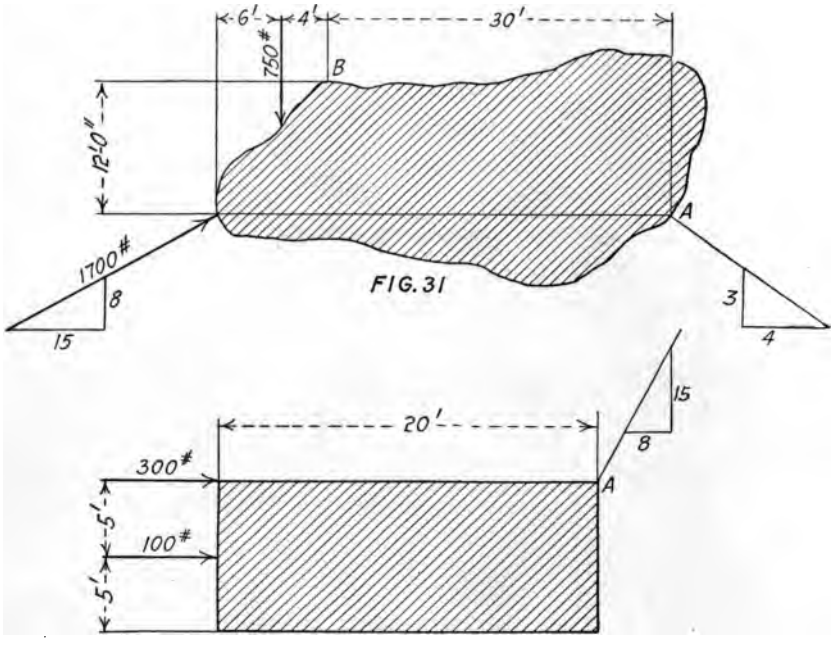


FIG. 31

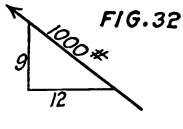


FIG. 32

find the magnitude of the force at A; and the direction of the force of 1,300 lb. at B.

7. Find the weight of the body in Fig. 32; the magnitude of the force at A; and locate the force of 1,000 lb. for equilibrium.

8. How far is the ball (Fig. 33) from the point A when the body is in equilibrium; what is the magnitude and direction of the force at A? Neglect the weight of the body.

9. The body in Fig. 34 weighs 1,000 lb. and resists two forces at C as indicated. The reactions at A and B are 820 lb. and 1,130 lb. respectively. In what direction does each reaction act and where is the vertical gravity line of the body?

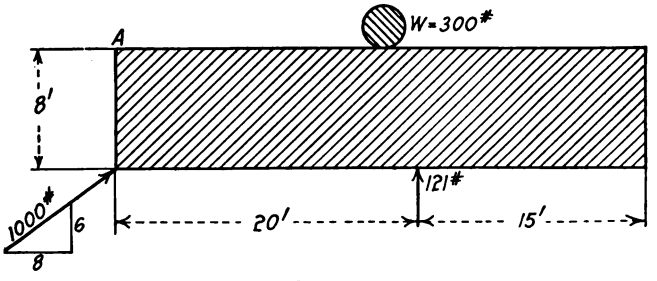


FIG. 33.

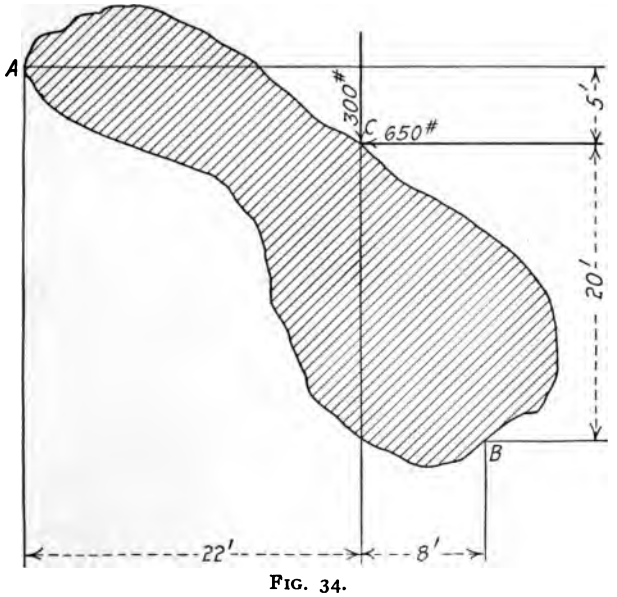


FIG. 34.

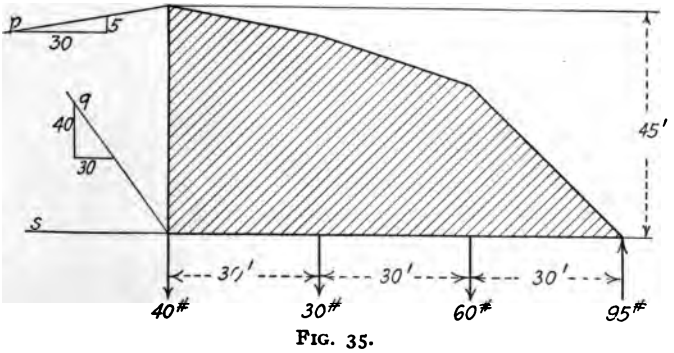


FIG. 35.

10. Neglect the weight of the body (Fig. 35) and find the magnitudes of the forces p, q and s.

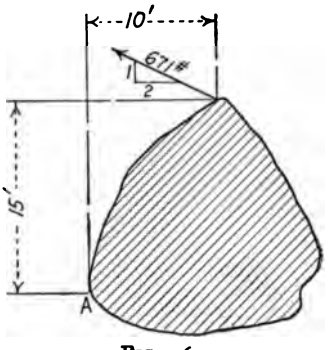


FIG. 36.

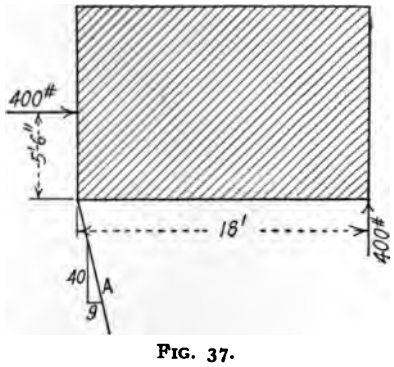


FIG. 37.

11. Find the weight and vertical gravity line of the body (Fig. 36) if the magnitude of the force at A is 781 lb. What is the direction of the force at A?

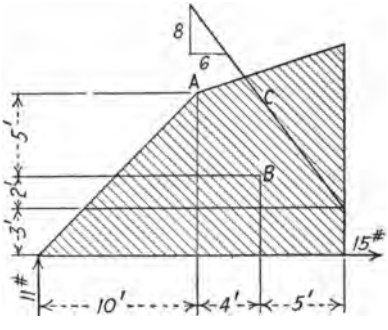


FIG. 38.

12. Neglect the weight of the body (Fig. 37) and find the magnitude of A, and the location and direction of a force of 1,220 lb. for equilibrium.

13. A level beam of uniform cross-section 10 ft. long, weighing 250 lb. and supported at each end, carries a single load of 5 tons at mid-span. Why is the problem of finding the two reactions indeterminate?

An algebraic solution for each of the two following problems is long and involved. Show that the graphic solution is comparatively short in each case.

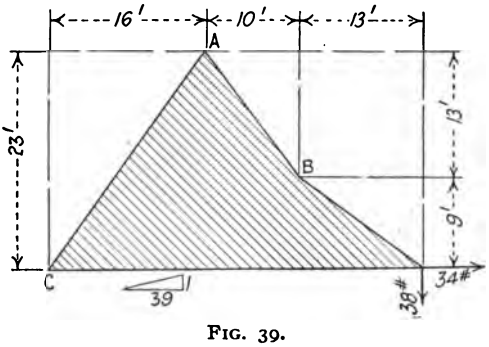


FIG. 39.

14. The magnitudes of the forces at A and B (Fig. 38) are 13 lb. and 17 lb. respectively. Neglect the weight of the body and find the directions of the forces at A and B, and the magnitude of the force C.

15. The magnitudes of the forces at A, B and C (Fig. 39) are 41 lb., 50 lb. and 25 lb. respectively. Neglect the weight of the body and find the three unknown directions.

CHAPTER II

APPLICATION OF THE PRINCIPLES OF EQUILIBRIUM

SEC. I. SIMPLE TRUSSES

37. Rigid Body.—In the foregoing section reference is frequently made to a "rigid body." No such thing exists in nature, if "rigid" is used in the sense of "unyielding." All solids may be considered to approximate in a greater or less degree an ideal state of complete rigidity. A steel casting, a granite rock, an oak block, putty and rubber—all illustrate different degrees of rigidity, for all change their shape under different degrees of pressure. In engineering parlance a rigid body is one which, in form and content, offers a sufficient resistance to distortion to give service as an engineering structure. Or, in other words, if the body is a solid, as an oak beam or steel-plate girder, consideration is given to the quality and quantity of the material in order that a sufficient degree of rigidity may be realized; but if the body is a framed structure, attention must *also* be given to the manner in which the structure is framed in order that it may be stable, or not be susceptible to an immediate and complete collapse. The frame represented in Fig. 21*h* is unstable or collapsible; for it offers no resistance to a change in its shape, except possibly a small amount of friction at the joints. The shape of the frame may be changed at will without affecting a change in the length of any member, and the same is true of any similar frame having four or more members.

A triangular frame differs from all others in this respect, since a change in its shape necessitates a change in the length of at least one of its three members. Hence a frame, which in outline presents a collection of triangles, is stable; and if it has adequate material in its members, it possesses a sufficient degree of rigidity for engineering purposes. Hence:

38. A Truss is a frame or jointed structure composed of members (usually straight) so connected as to form a succession of triangles making one rigid or stable structure. Acting as a whole, this performs the functions of a beam in resisting distortion[1] caused by shearing forces and bending moment; while the individual members are designed (except in special cases) to perform the functions of a tie or strut in resisting changes in their length caused by tensile or compressive forces.

39. Stability.—The simplest truss that can be constructed has three members m, and three joints j. A more elaborate truss comprises several triangular frames, so combined that each additional triangle adds one joint and two members. Hence, the number of members necessary to insure stability under any arrangement of loading is

$$m = 2j - 3 \qquad (1)$$

A truss having fewer members than are required by Eq. (1) is in a state of unstable equilibrium, and will collapse except under special conditions of loading. A truss having more members than are indicated by Eq. (1) is a redundant structure. Such structures, if the members are properly placed, will support loads of any arrangement. In some instances this is a distinct advantage. Redundant structures cannot be analyzed by the principles of statics alone, and are said to be "statically indeterminate." By giving due consideration to the elastic properties of the members; or by the aid of certain reasonable assumptions which have the sanction of good engineering practice, a satisfactory analysis may be made.

40. The assumption that the members of a truss are subject to tensile and compressive forces only, presupposes four conditions:

[1] It should be clearly understood and constantly kept in mind that structures are not rigid in the sense of unyielding, but are elastic in the same sense that a rubber eraser or a rubber band is elastic, only in a lesser degree. The members of a steel truss, designed in accordance with current specifications, will lengthen or shorten from $\frac{1}{32}$ to $\frac{1}{16}$ of an inch for every 10 ft. of length when the structure is supporting the full load for which it was designed. This change in length inevitably changes the shape of the triangular units of the truss and accounts for the distortion of the whole structure. The distortion is comparatively so slight, however, that for our present consideration, the changes in the dimensions of the truss may be considered negligible without appreciable error.

1. That the axial gravity lines of the members meeting at a joint, intersect at one point.

2. That the loads are applied only at the joints.

3. That the members are subject to neither shear nor bending moment due to their own weight.

4. That the joints are frictionless hinges.

The first and second conditions are generally incorporated in a good design. The third condition is true of vertical members only. The nearest approximation to the fourth condition is found in a pin-connected truss.

41. The application of the principles of equilibrium to the analysis of a framed structure may be considered as two distinct operations: (*a*) the determination of the external forces; and (*b*) the determination of the internal forces.

42. The external forces acting on a structure are: (*a*) the loads which the structure supports; (*b*) the weight of the structure itself; and (*c*) the forces acting at the points of support, commonly called the reactions.

The loads are usually known or easily determined, but the weight of the structure not as yet designed must be assumed. The reactions are determined by a solution of a system of non-concurrent forces, as set forth in Sections V and VI of Chapter I. The external forces are generally analyzed by the algebraic method. This process is usually shorter than the graphic method, especially if the external forces are parallel.

43. The internal forces acting in the structure are many and of various kinds. We shall consider here only those of primary importance—the tensile and compressive forces acting along the members. These forces are known as stresses.

As the reactions are determined by the application of the principles of the equilibrium of coplanar forces to the structure as a whole, so the stresses in the members are determined by the same principles; but applied to isolated parts of the structure. The portion to be considered may be a single joint, or it may include several joints and members. The forces to be considered may represent reactions, loads or stresses. There are two principal methods of procedure in the determination of stresses in framed structures:

1. The method of joints
   (*a*) algebraic
   (*b*) graphic.
2. The method of sections
   (*a*) algebraic
   (*b*) graphic.

44. The Method of Joints.—The truss in Fig. 40*a* supports five loads as shown. What is the stress in each member if the reactions are vertical?

The determination of the stresses by the method of joints, either algebraically or graphically, is simply a solution of as many systems of concurrent forces as there are joints in the truss. After the external forces have been found, the solution begins at a joint where the stresses of two members concur with one or more external forces; proceeds to an adjacent joint which presents but two unknown stresses; and so on throughout the entire structure. The force exerted on a joint by one end of a member is equal in magnitude but opposite in sense to the force exerted by the member on the joint at its other end. The kind of stress in a member is indicated by the sense of the force which a member exerts on a joint; *i.e.*, the stress is tensile if the sense is away from the joint and compressive if toward the joint.

Graphic Solution.—The first step in the solution of any problem of this character is the determination of the forces necessary to place the truss in equilibrium, *i.e.*, to find the reactions. The left and right reactions, determined algebraically, are 50 lb. and 55 lb. respectively. It is a waste of time to find the reactions graphically when the external forces are parallel. A capital letter is placed in the space between each external force and in each triangle of the frame. The outline of the truss represents essentially a location-direction diagram, and we proceed to draw a closed magnitude-direction diagram for each of the six concurrent systems—one for each joint. Since the end joints present but two unknowns, either may be chosen for the drawing of the first diagram. The conventional method is to begin at the left end of a structure, proceeding in a clockwise rotation about each joint. All location-directions are known. The

problem is to determine unknown magnitudes or stresses acting in the various members.

Beginning at the left-end joint, proceeding in a clockwise rotation, and including first the location-directions having known

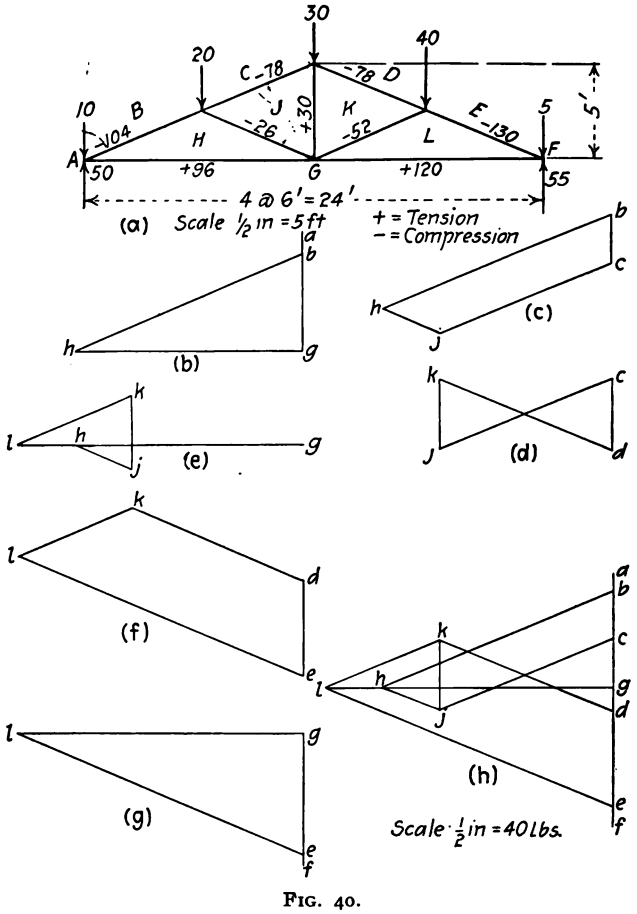


FIG. 40.

magnitudes, we read *GA*, *AB*, *BH* and *HG*. The magnitude-directions *ga*, *ab*, *bh* and *hg* are laid off in the same order in Fig. 40*b*. The magnitude-directions *ga* and *ab* are known, and the diagram is closed by drawing *bh* and *hg*. The sense *bh* is toward the joint; the sense *hg* is away from it. The magnitude-direction *bh* scales 104 lb.; the magnitude-direction *hg* scales 96 lb.;

hence the stresses in *BH* and *HG* are 104 lb. compressive, and 96 lb. tensile respectively. These numerical values, with minus and plus signs to signify compression and tension, are indicated on the members of the truss.

The adjacent upper joint now presents only two unknown stresses. The known magnitude-directions *hb* and *bc* are laid off (Fig. 40*c*) and the diagram closed by drawing *cj* and *jh*; giving −78 lb. and −26 lb. for the stresses in the members *CJ* and *JH*.

At the lower middle joint the stresses in three members *JK*, *KL* and *LG* are unknown and a solution is not yet possible; but at the upper middle joint the stresses in only two members are unknown, and the solution is given in Fig. 40*d*.

There are but two unknowns at each of the three remaining joints. Fig. 40*e* represents the diagram for the lower middle joint, and Fig. 40*f*, the joint supporting the load of 40 lb. The stresses in all the members have been determined, and all the forces concurring at the right-end joint are known; but the diagram (Fig. 40*g*) is drawn for this joint as a check on the foregoing solutions.

Any two adjacent joints have one member in common, consequently their respective magnitude-direction diagrams also have one side in common; hence, any one of the diagrams may be added to the one preceding or following it. Thus one figure may be developed which will contain all the magnitude-direction diagrams representing graphically the magnitude, direction and sense of each external force and internal stress of the structure. Such a figure is called a *stress diagram* and is illustrated in Fig. 40*h*. The magnitude-direction diagram of the external forces or load line *ab*, *bc*, *cd*, *de*, *ef*, *fg* and *ga* is drawn *first* and the lines representing the stresses follow in regular order. Beginning at the left-end joint, we read *ga*, *ab* and draw *bh* and *hg* parallel respectively to *BH* and *HG*. For the next joint we read *hb*, *bc* and draw *cj* and *jh*. For the top joint we read *jc*, *cd* and draw *dk* and *kj*. For the bottom joint we read *gh*, *hj* and *jk*, and draw *kl* and *lg*. For the next joint we read *lk*, *kd* and *de* and draw *el* parallel to *EL*. If this line, drawn from *e* parallel to *EL* passes through the point *l*, we have a check

on our work, the stress diagram closes and the solution is complete.

When the external forces are parallel, as in the present case, the magnitude-direction diagram for the external forces is a straight line. The stress diagram is essentially a magnitude-direction diagram for the external forces and a magnitude-direction diagram for the concurrent system at each joint.

Algebraic Solution.—The concurrent systems are sketched in Fig. 41. The V-component of a (Fig. a) is 40 lb. acting downward. The H-component is $\frac{12}{5} \times 40 = 96$ lb. acting to the left; hence, $b = 96$ lb. acting to the right or away from the joint; $a = \frac{13}{5} \times 40 = 104$ lb. acting toward the joint. Assume H- and V-components for c and d as shown in Fig. b.

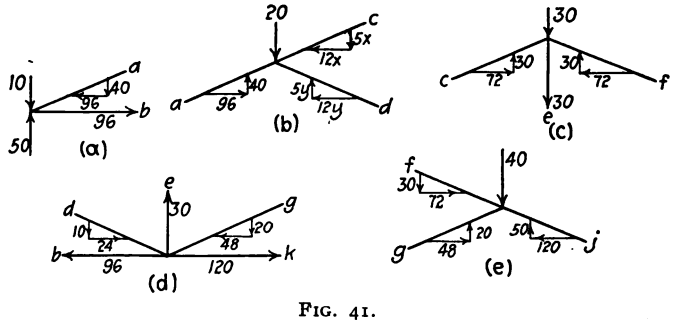


FIG. 41.

$$\begin{aligned} 5x + 20 &= 5y + 40 \\ 12x + 12y &= 96 \\ \hline x &= 6 \\ y &= 2 \end{aligned}$$

H-component of $c = 72$ lb. acting to the left.
V-component of $c = 30$ lb. acting downward.
$c = 78$ lb. acting toward the joint.
H-component of $d = 24$ lb. acting to the left.
V-component of $d = 10$ lb. acting upward.
$d = 26$ lb. acting toward the joint.

The upper middle joint is sketched in Fig. c. The H-component of $c = 72$ lb. acting to the right, consequently the H

component of $f = 72$ lb. acting to the left. Hence, the V-component of $f = 30$ lb. acting upward; $f = 78$ lb. acting toward the joint; and $e = 30$ lb. acting downward or away from the joint, to balance the load of 30 lb. and the V-components of c and f.

At the lower middle joint (Fig. d) the V-component of $g = 20$ lb., acting downwardt o balance e and the V-component of d; therefore the H-component of $g = 48$ lb. acting to the left, and $g = 52$ lb. acting toward the joint. Hence, $k = 120$ lb., acting away from the joint.

In Fig. e there is but one unknown force j. The H-component $= 120$ lb. acting to the left, the V-component $= 50$ lb. acting upward and $j = 130$ lb. acting toward the joint. We now have a check upon our computations by noting that the V-component of j equals the right reaction, minus the load of 5 lb.; the H-component equals k; and the ratio of the V- and H-components of j is $\frac{50}{120}$ or $\frac{5}{12}$, which is the slope or bevel of the member.

Assuming that the stresses in BH and CJ are unknown, solve algebraically and graphically for the stress in JH, using the method of joints.

Remove the loads of 20 lb. and 40 lb., and determine the stresses in the three web members HJ, JK and KL.

The graphic solution is generally employed when the stresses are determined by the method of joints. This is especially true in the case of roof trusses having inclined chords. There is perhaps little preference between the algebraic and the graphic solutions, if the top and bottom chords are horizontal and half the web members are vertical. There are several factors which conspire in making the method of joints peculiarly efficient in truss analysis for *stationary* loads.

1. The stress in any member is constant for a given load.
2. The required stresses in all members occur simultaneously.
3. The external forces and required stresses concurring at a joint are in equilibrium.

The stress in any member of a truss supporting *moving* loads varies as the loads move upon the structure. The stress in any member is desired only for that particular position of the loads

which will cause the maximum stress in that member. Some members receive their maximum stress when the moving loads cover the entire structure, while in other members the stress is greatest when only a portion of the structure is covered. *The maximum stresses do not occur simultaneously.* This fact greatly limits the usefulness of the stress diagram in the case of moving loads. For example, suppose that four members A, B, C and D meet at a joint; and that the moving loads are in the position for maximum stress in B. A stress diagram will give the stresses not only in B; but in A, C, D and all the other members of the structure as well. The stress in B, however, is the only stress of any value which may be obtained from that particular stress diagram; since it is the only member in which the stress is a maximum.

45. The method of sections gives excellent service when the stress in but one member is desired for any particular arrangement of loads. The algebraic solution is generally preferred. Suppose we wish to determine the stress in the member GH (Fig. 42*a*). If the member were cut or removed from the truss, it is perfectly obvious that the part ACG would rotate clockwise, and the part CMH would rotate counter-clockwise—each about the pivot C; the joints G and H moving further apart. Hence in preserving the equilibrium of the structure, it is plainly the duty of the member GH to hold the joint G from moving to the left, and the joint H from moving to the right. In performing this function the member exerts a force to the right on the joint G, and an equal and opposite force to the left on the joint H.

Imagine that a plane of section XY cuts the three members CD, CH and GH, dividing the truss into two portions. If p, q and s represent the stresses in the members before cutting, then the known external forces (loads and reactions) acting upon each portion are balanced by the forces p, q and s as illustrated in Figs. 42*b* and *c*. The left portion (Fig. 42*b*) represents a coplanar system of five non-concurrent forces having three unknown magnitudes; corresponding to the first combination listed in Article **34**. An algebraic and graphic solution for this system is given in Article **35**, Problem 4.

The algebraic solution when performed by writing and solving three simultaneous equations is known as Rankine's[1] "Method of Sections." The corresponding graphic solution is the work of Culmann.[2] The algebraic solution, whereby any one of the three unknown magnitudes is determined by balancing the moments of all the forces about the intersection of the other two forces having unknown magnitudes, is known as Ritter's[3] "Methods of Moments." In order to determine the stress by Ritter's method, we balance the moments of all the external forces about C (Figs. 42*b* or *c*) thereby eliminating the two unknown magnitudes p and q.

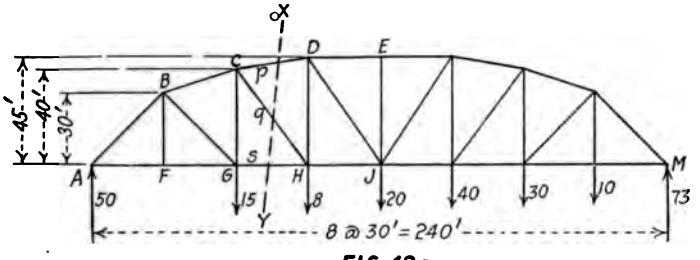


FIG. 42a

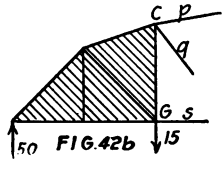


FIG. 42b

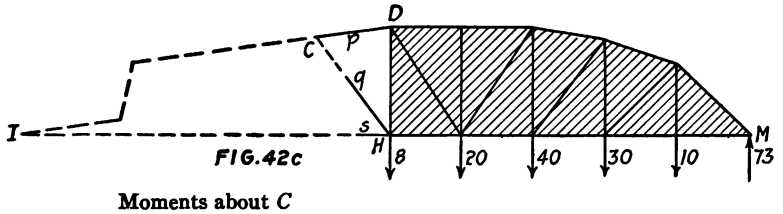


FIG. 42c

Moments about C

FIG. 42*b*	FIG. 42*c*	
$60 \times 50 = 3{,}000$	$30 \times 8 = 240$	$180 \times 73 = 13{,}140$
	$60 \times 20 = 1{,}200$	$10{,}140$
	$90 \times 40 = 3{,}600$	$3{,}000$
	$120 \times 30 = 3{,}600$	
	$150 \times 10 = 1{,}500$	
	$10{,}140$	

[1] Applied Mechanics.
[2] Die Graphische Statik. 1886.
[3] Dach- und Brucken constructionen. 1862.

The sum of the moments about C of the known forces acting on the left-hand portion is 3,000 ft.-tons clockwise; hence, the force s acting at a distance of 40 ft. from C must have a magnitude of 75 tons toward the right or away from the joint G to balance the moments. The sum of the moments about C of the known forces acting on the right-hand portion is also 3,000 ft.-tons, but counter-clockwise; hence, the force s acting at a distance of 40 ft. from C must have a magnitude of 75 tons toward the left, or away from the joint H to balance the moments.

The fact that the sum of the moments about C of the known external forces is the same (3,000 ft.-tons) for either portion, except that the rotation is clockwise in one case and counter-clockwise in the other, is easily explained. The truss (Fig. 42*a*) is in equilibrium and the sum of the moments of the eight known external forces about any point C, for example, equals zero. If these eight forces are divided in any manner into two groups, the sum of the moments of one group about C must balance the sum of the moments of the other group.

Determine p and q (Fig. 42*c*) by balancing the moments of the forces about the points H and I respectively, and note that the results check with the computations of Problem 4, Article 35.

Since the unknown magnitudes may be determined by balancing the moments of the forces on *either* side of the section, it is expedient to consider the side which has the fewer forces.

The stress in DJ (Fig. 42*a*) may be determined by passing a section through DE, DJ and HJ. The two chord members are parallel and there is no point of their intersection about which the moments may be balanced; but since the chords are horizontal, the vertical component of the stress in DJ must balance the vertical magnitudes on either side of the section. The resultant of the vertical forces on the left of the section is $50 - (15 + 8) = 27$ tons acting upward. The left-hand portion will move upward, unless the vertical component of DJ is 27 tons acting downward from D or away from the joint. Therefore the member is in tension and the vertical component of the stress is 27 tons.

Compute the stresses in *DE* and *HJ* and see if these stresses balance with the horizontal component of *DJ*.

Compute the stresses in *BC*, *BG*, *CG* and *DH* and then draw a stress diagram to check the computations.

It is worthy of note that a solution by the method of sections is entirely independent of the number, inclination or arrangement of any members of the structure other than those cut by the section.

46. Problems.

1. The truss in Fig. 43 is supported by vertical reactions at each end. Deter-

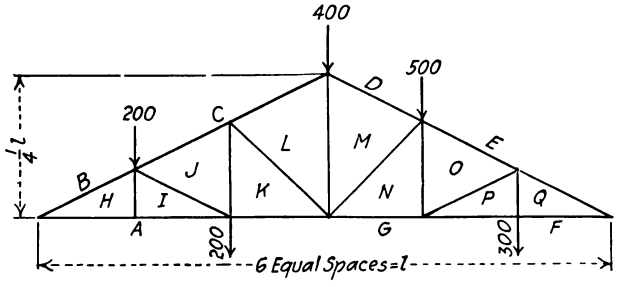


FIG. 43.

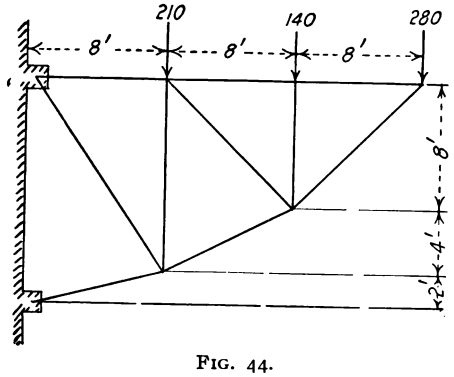


FIG. 44.

mine the reactions algebraically, construct the magnitude-direction diagram for the external forces (sometimes called the load line) and draw a stress diagram. Mark the stresses with proper signs upon the members. Note that *h* and *i* fall at the same point in the stress diagram, indicating that the member *HI* has no stress and is superfluous for this particular loading. Note also that the stresses in *QF* and *GP* are equal, and that the stress in *PQ* equals the external load at the joint. In consideration of these observations develop a general rule by

which superfluous members may be eliminated before the stress diagram is begun.

2. The frame in Fig. 44 has been substituted for the solid body of Fig. 25*a*. Draw a stress diagram. Note that the magnitude-direction diagram for the external forces, which constitutes the beginning or foundation of the stress diagram, has already been constructed in Fig. 25*d*.

3. Substitute a frame for the solid body in Fig. 30 and draw a stress diagram.

4. The truss in Fig. 45 is supported by vertical reactions at *A* and *B*. Draw a stress diagram. Compute the stresses in *X*, *Y* and *Z*.

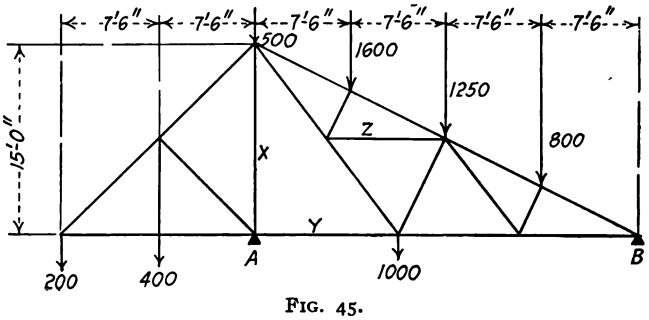


FIG. 45.

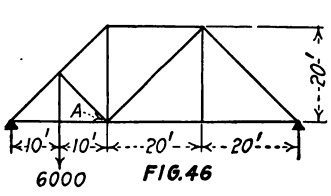


FIG. 46

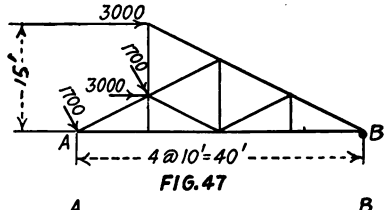


FIG. 47

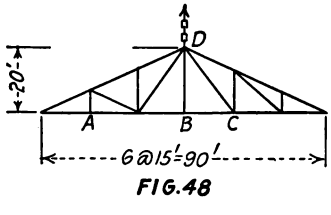


FIG. 48

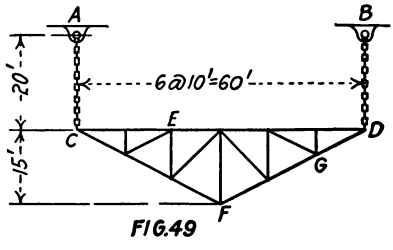


FIG. 49

5. Compute the *H*- and *V*-components of the stresses in the five members meeting at *A* (Fig. 46). Mark the values on a sketch and indicate whether the members are in tension or compression.

6. The truss in Fig. 47 is supported at *A* and *B*. The roller at *B* indicates a vertical reaction. Draw a stress diagram.

7. The truss in Fig. 48 is supported by a chain at *D*. Suspend weights of 150 lb. at *A*, 50 lb. at *B* and 50 lb. at *C*, and draw a stress diagram.

8. The truss in Fig. 49 is supported by chains *AC* and *BD*. Draw a stress diagram for vertical loads of 300 lb. at *E*; 600 lb. at *F*; and a horizontal load of 450 lb. at *G*, acting to the right.

SEC. II. STRUCTURES REQUIRING SPECIAL CONSIDERATION

47. The application of the principles of static equilibrium to the framed structures, discussed in the preceding section, presented no real difficulties. There are types of structures, however, for which the solutions are not so simple; and a special investigation is necessary in order that the difficulties encountered may be overcome. These difficulties are caused by indeterminate reactions, indeterminate stresses or both. Statically indeterminate structures *as such* will not be treated here. There are, however, several indeterminate types which may be treated by static methods; when certain reasonable assumptions which have the sanction of good engineering practice are made. There are also several types which are indeterminate in appearance but not in fact.

48. The Fink Truss.—The structure outlined in Fig. 50*a*, and known in this country as a Fink truss, is a good example of a case in which the stresses are apparently statically indeterminate. A difficulty is encountered after the stress diagram (Fig. 50*b*) has been drawn for joints 1, 2 and 3; for the stresses in three members at each of the joints 4 and 5 are unknown. In this emergency, one of two graphic solutions (*a*) or (*b*) may be used.

(*a*) Temporarily remove the members KL and LM, substitute the dotted member OM connecting the joints 5 and 7, and draw the polygon *jhdeoj* for the joint 4. Then the polygon *oefmo* for joint 7 may be drawn, giving the stress in the member FM; which is in no way influenced by the arrangement of members in the quadrilateral 4-5-6-7. Remove the dotted member and replace the members KL and LM. The point m is now located in the stress diagram; the stress in FM is known; and the remaining stresses are easily determined by taking the joints in the order 7-4-5-6.

(*b*) The following solution is based upon the exception to the general law of concurrent forces laid down in Article **17.** The point j was located in the stress diagram when joint 3 was solved. The point k will fall somewhere on the line through j parallel to KJ. The stress in ML can be determined since the

members LE and FM are in the same straight line. Assume any value for the stress in the member LE, as $l'e$; and draw the stress polygon $l'efm'l'$, giving $m'l'$ for the stress in ML. Since there remain but three members having unknown stresses at joint 6, and two of the members are in the same straight line, the stress polygon $l'm'n'k'l'$ can be drawn and the stress in KL determined.

Any other value $l'e$ might have been assumed for the stress in the member LE; and for each different point l' on the line

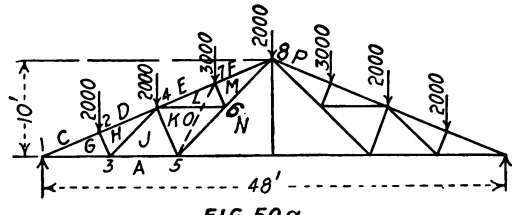


FIG. 50a

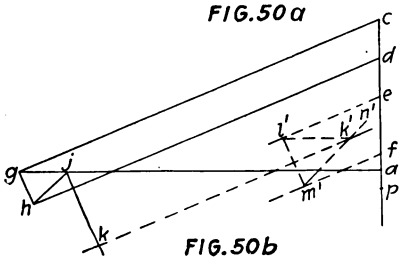


FIG. 50b

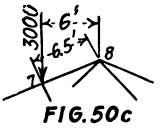


FIG. 50c

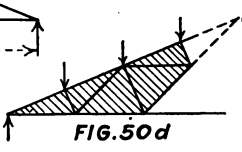


FIG. 50d

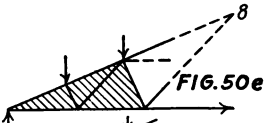


FIG. 50e



FIG. 50f

through a parallel to LE, there is a corresponding point k' on a line parallel to LE. Hence the point k is located at the intersection of the line through j parallel to KJ, with the line through k' parallel to LE. The stresses can now be solved for either joint 4 or 5 and the diagram completed without any further difficulty.

Complete the stress diagram for solutions (*a*) and (*b*).

When computing the stresses by the method of sections, we meet with a difficulty somewhat similar to the one encountered in drawing the stress diagram. While it is impossible to divide the truss by a section cutting any one of the members JK, KN, KL, LE or ML without cutting more than three members; yet the stresses in these members are easily determined when

taken in the proper order. The stress in the member *ML* may be determined by a section cutting out the portion shown in Fig. 50*c*. All of the five members which are cut, except the member *ML*, intersect at the peak; about which point the moments of the load at joint 7 and the stress in the member *ML* must balance. The stress in the member *NA* may be determined by balancing the moments of all the forces about the joint 8 (Fig. 50*d*). After the stress in the member *NA* has been determined and indicated in Fig. 50*e*, the stress in the member *KL* may be found by balancing the moments of all the forces about the joint 8. When the stress in the member *KL* has been indicated in Fig. 50*f*, the stress in the member *JK* may be determined by balancing the moments of all the forces about the joint 1. After the stress in the member *KL* has been indicated in Fig. 50*e*, the stress in *LE* may be determined by balancing the moments of all the forces about the joint 5.

Compute the stresses in the members mentioned above and see if they check with the stress diagrams.

Formulate a general rule for applying the method of sections which will cover all cases which have heretofore been considered.

49. The three-hinged arch has three pin-connected joints—one at each of its two points of support and a third usually located at mid-span (Fig. 51*a*). The structure, being free to turn at points *A*, *B* and *C*, will obviously collapse even under vertical loads unless the sustaining forces at *A* and *B* have horizontal as well as vertical components. Hence four unknown elements for equilibrium must be determined—the magnitude and direction of each reaction. Since only three independent equations can be written for a non-concurrent system of forces acting upon a rigid body, the problem of finding four unknown elements is apparently impossible. Such, however, is not the case. The problem is rendered statically determinate by the condition that the structure is composed of two rigid bodies, *X* and *Y*, each free to turn about two points.

The horizontal magnitudes of the external forces acting on the structure as a whole may balance, the corresponding vertical magnitudes may balance, and the moments about *C* of the external forces acting on *X* may balance; but these three condi-

tions do not certify that the moments about C of the external forces acting on Y are balanced. This condition provides the possibility of making four independent statements or equations concerning the equilibrium of the structure.

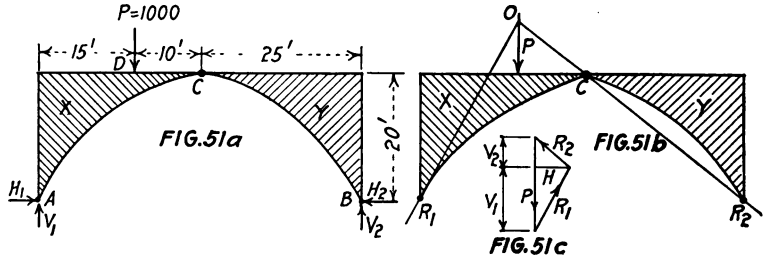
Let us consider the case of a vertical load $P = 1{,}000$ lb. at D.

Algebraic Solution.—Assume V_1 and H_1 for the components of the reaction at A; and V_2 and H_2 for the components of the reaction at B. Balance the moments of all the external forces acting on the structure about A.

$$V_2 = \frac{1{,}000 \times 15}{50} = 300$$

Similarly

$$V_1 = \frac{1{,}000 \times 35}{50} = 700$$



The moment of V_2 about C is

$$25 \times 300 = 7{,}500 \text{ ft.-lb.}$$

and the body Y will rotate counter-clockwise about C, unless the moment of V_2 is balanced by the moment of H_2; hence

$$H_2 = \frac{7{,}500}{20} = 375 \text{ lb., acting to the left.}$$

The algebraic sum of the moments of P and V_1 about C is

$$25 \times 700 = 17{,}500$$
$$10 \times 1{,}000 = 10{,}000$$
$$7{,}500 \text{ ft.-lb., clockwise.}$$

The body X will rotate clockwise about C unless the moments of P and V_1 are balanced by the moment of H_1; hence

$$H_1 = \frac{7{,}500}{20} = 375 \text{ lb., acting to the right.}$$

The magnitude of H_1 could have been determined by balancing the horizontal magnitudes after the magnitude of H_2 has been found. The magnitude and direction of each reaction is readily determined from the magnitudes of its components.

Graphic Solution.—There are three external forces P, R_1 and R_2 (Fig. 51*b*) acting upon the structure. The reaction R_2 obviously acts through C, for otherwise the body Y would rotate about C. This condition determines the location-direction of R_2 which intersects the force P at O; through which point the location-direction of R_1 must pass if there is to be no rotation of the structure about the point O. The magnitude-direction diagram for the concurrent system at O is drawn in Fig. 51*c*, giving the magnitudes of R_1 and R_2.

In the case where both portions X and Y are supporting one or more loads, the reactions for each load may be determined separately after the manner just described. The reactions at each support thus found may then be combined and the resultant reaction for all loads determined. A more complicated but somewhat shorter solution is given in the next article.

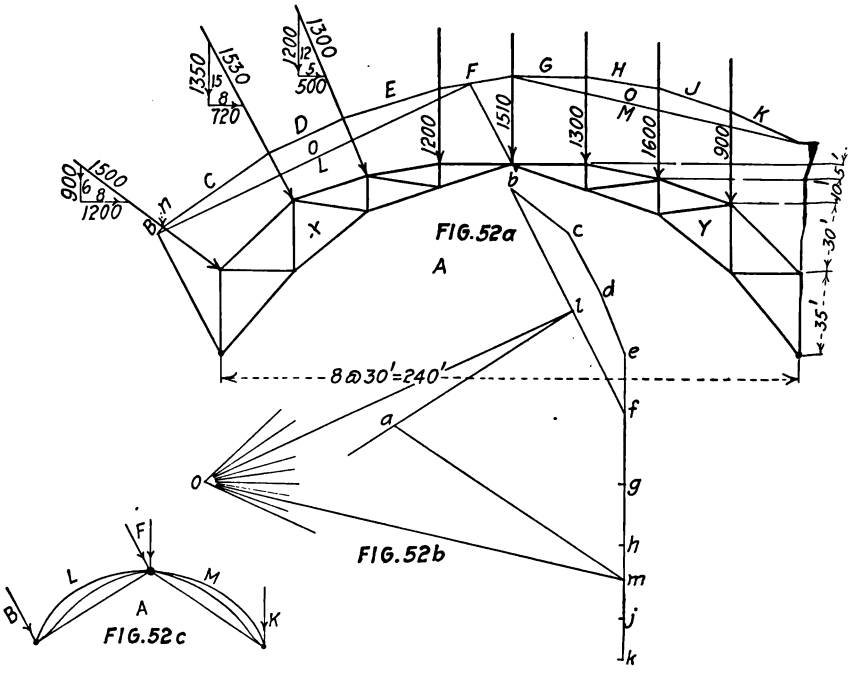
50. Illustrative Problem.—The three-hinged arch (Fig. 52*a*) supports eight loads. Determine the reactions.

Graphic Solution.—Lay off the magnitude-directions of the eight loads from *b* to *k* (Fig. 52*b*). Choose any convenient pole *o* and imagine that the nine magnitude-directions of the components *ob* to *ok* are drawn. Choose any convenient point *n* on the location-direction *BC*, and draw the location-directions of the components *OB* to *OK*.

Draw the magnitude-direction *bf;* and through the left and center hinges draw lines parallel to *bf*, intersecting the location-directions *OB* and *OF*. Connect these intersections by the location-direction *OL*, which closes the location-direction diagram for the four forces *B* to *F*. Draw the magnitude-direction *ol*, which determines the magnitudes of two loads *bl* and *lf;* which, if applied at the left and center hinges respectively, will have the same effect upon the equilibrium of the body *X* as the four loads *B* to *F*.

The magnitude-direction of the resultant of the four loads

F to K is fk. Through the center and right hinges draw line parallel to fk intersecting the location-directions OF and OK. Connect these intersections with the location-direction OM, closing the location-direction diagram for the four forces F to K. Draw the magnitude-direction om, which determines the magnitudes of two forces fm and mk; which, if applied at the center and right hinges respectively, will have the same effect



upon the equilibrium of the body Y as the four loads F to K.

For the sake of clearness the location-directions BL, LF, FM and MK are reproduced in Fig. 52c. Two forces are acting at the left hinge—the reaction AB and the load BL; the location-direction of their resultant AL must pass through the left and center hinges. Therefore through l draw the magnitude-direction parallel to the location-direction AL. Similarly the location-direction of the resultant of the load MK and the reaction KA acting at the right hinge is MA, which must pass through the center and right hinges. Hence through m

draw the magnitude-direction parallel to the location-direction MA intersecting the magnitude-direction through l at a.

The location of a fixes the magnitude-directions ka and ab which, when drawn, will close the magnitude-direction diagram for the external forces and completely determine the reactions.

Draw the stress diagram and note that the diagram gives a check on itself when the center hinge is reached.

It may be well to note that the load FG acting at the center hinge, which was combined with the loads on the body Y, might have been combined with the loads on the body X without affecting the final results.

Prove the statement in the preceding paragraph by a solution.

Algebraic Solution.—Replace each inclined load by its H- and V-components. Assume V_1 and H_1 for the components of the left reaction; and V_2 and H_2 for the components of the right reaction. Balance the moments of all the forces acting on the structure about the left support.

$$
\begin{aligned}
35 \times 1{,}200 &= 42{,}000 \\
65 \times 720 &= 46{,}800 \\
75 \times 500 &= 37{,}500 \\
30 \times 1{,}350 &= 40{,}500 \\
60 \times 1{,}200 &= 72{,}000 \\
90 \times 1{,}200 &= 108{,}000 \\
120 \times 1{,}510 &= 181{,}200 \\
150 \times 1{,}300 &= 195{,}000 \\
180 \times 1{,}600 &= 288{,}000 \\
210 \times 900 &= 189{,}000 \\
240)\,1{,}200{,}000 & \\
5{,}000 &= V_2
\end{aligned}
$$

Balance the moments about the right support,

$$
\begin{aligned}
30 \times 900 &= 27{,}000 \\
60 \times 1{,}600 &= 96{,}000 \\
90 \times 1{,}300 &= 117{,}000 \\
120 \times 1{,}510 &= 181{,}200 \\
150 \times 1{,}200 &= 180{,}000 \\
180 \times 1{,}200 &= 216{,}000 \\
210 \times 1{,}350 &= 283{,}500 \\
240 \times 900 &= 216{,}000 \\
1{,}316{,}700 & \\
126{,}300 & \\
240)\,1{,}190{,}400 & \\
4{,}960 &= V_1
\end{aligned}
$$

$$
\begin{aligned}
75 \times 500 &= 37{,}500 \\
65 \times 720 &= 46{,}800 \\
35 \times 1{,}200 &= 42{,}000 \\
126{,}300 &
\end{aligned}
$$

Balance the moments of the forces on the body Y about the center hinge,

$30 \times 1{,}300 = 39{,}000$
$60 \times 1{,}600 = 96{,}000$
$90 \times 900 = 81{,}000$
$216{,}000$

$120 \times 5{,}000 = 600{,}000$
$216{,}000$
$80)384{,}000$
$4{,}800 = H_2$
acting to the left.

Balance the moments of the forces on the body X about the center hinge,

$30 \times 1{,}200 = 36{,}000$
$60 \times 1{,}200 = 72{,}000$
$5 \times 500 = 2{,}500$
$90 \times 1{,}350 = 121{,}500$
$15 \times 720 = 10{,}800$
$120 \times 900 = 108{,}000$
$45 \times 1{,}200 = 54{,}000$
$404{,}800$

$120 \times 4{,}960 = 595{,}200$
$404{,}800$
$80)190{,}400$
$2{,}380 = H_1$
acting to the right.

Check by balancing the horizontal magnitudes,

$2{,}380$
$1{,}200$
720
500
$4{,}800 = 4{,}800$

Compare these results with the H- and V-components of ka and ab (Fig. 52*b*).

51. The cantilever bridge supporting vertical loads and resting upon four vertical supports (Fig. 53*a*) appears to be

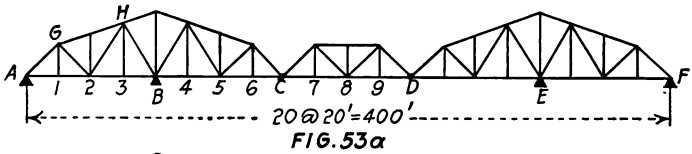


FIG. 53a

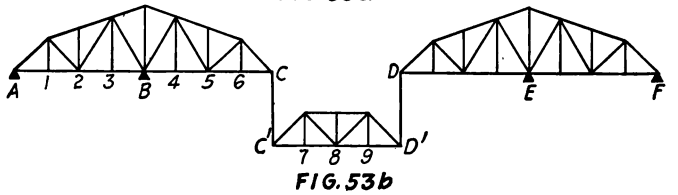


FIG. 53b

statically indeterminate, for there are four magnitudes to determine—one at each support; but the indeterminateness disappears when we consider that the structure is composed of three rigid bodies AC, CD and DF fastened together by pin-connected joints at C and D. The portions AC and DF are

called cantilever trusses; AB and EF are the anchor arms and BC and DE are the cantilever arms. The portion CD is called the center or suspended truss. The interaction of one part of the structure upon another is seen more clearly by a consideration of Fig. 53*b*, where the center truss is shown as if it were suspended from the cantilever arms.

Loads applied at points 1, 2 and 3 are supported at A and B; and that part of the structure represented by $AGHB$ performs the function of a simple truss. Loads at points 4, 5 and 6 are also supported at A and B; but in this case the reaction at A is negative, *i.e.*, acts downward. Consequently, the structure must be provided with anchors into the masonry at A and F.

Suppose that a load of 1,000 lb. is placed at point 7. There is a tension of 750 lb. in CC' causing a negative reaction of 750 lb. at A, and an upward pressure of 1,500 lb. at B. Similarly there is a negative reaction of 250 lb. at F, and an upward pressure of 500 lb. at E. The reactions at E and F for loads between D and F are determined in the same manner as for the left cantilever truss.

The foregoing analysis as applied to the structure in Fig. 53*b* is equally applicable in Fig. 53*a*.

52. Inclined Loads.—In the next chapter we shall have occasion to consider the effect of loads acting normal to the

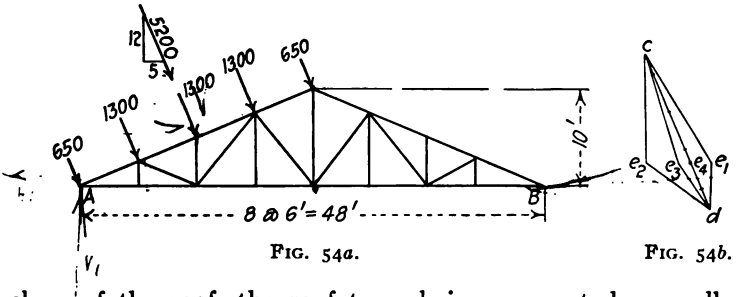


FIG. 54*a*.

FIG. 54*b*.

slope of the roof, the roof truss being supported on walls, as illustrated in Fig. 54*a*. The resultant of the five loads is 5,200 lb. Its H- and V-components are 2,000 lb. and 4,800 lb. respectively.

Let H_1 and V_1 represent the H- and V-components at A, and H_2 and V_2 represent the corresponding components at B. Since

the points of support are level and the H-components act in a straight line, V_1 and V_2 may be determined independently of H_1 and H_2.

Balance the moments of all the forces about the point A,

$$\frac{5{,}200 \times 13}{48} = 1{,}408 = V_2$$

$$4{,}800$$

$$1{,}408$$

$$3{,}392 = V_1$$

Check the value of V_1 by balancing the moments of all the forces about B.

The magnitudes of H_1 and H_2 acting in a straight line cannot be determined (as in the case of a three-hinged arch consisting of two rigid bodies) by the principles of statics. Their sum is 2,000 lb., and the magnitude of each depends somewhat upon the manner in which the truss is supported. If the truss has a roller support at B and we assume no friction, then

$$H_1 = 2{,}000$$

and $$H_2 = 0$$

If the truss has a roller support at A and we assume no friction, then

$$H_1 = 0$$

and $$H_2 = 2{,}000$$

If we assume that the horizontal thurst is resisted equally at A and B, then

$$H_1 = 1{,}000$$

and $$H_2 = 1{,}000$$

If we assume that the horizontal component of each reaction is proportional to its vertical component, then

$$H_1 = \frac{3{,}392}{4{,}800} \times 2{,}000 = 1{,}413$$

and $$H_2 = \frac{1{,}408}{4{,}800} \times 2{,}000 = 587$$

In the latter case the resultant of the H- and V-components at each support has a direction parallel to the resultant of the loads.

In Fig. 54*b* the load line *cd* is laid off and the magnitude-

direction diagram 1or the external forces is closed from d to c in four different ways; corresponding to the various assumptions which may be made regarding the magnitudes of H_1 and H_2. Since the vertical magnitudes are constant for any assumed magnitudes for H_1 and H_2, the points e_1, e_2, e_3 and e_4 fall on a horizontal line.

Complete the stress diagram. How do the different assumptions affect the stresses?

SEC. III. BEAMS: SHEAR AND BENDING MOMENT DIAGRAMS

53. A beam is any part of a structure or the structure itself as a whole, which resists the action of transverse forces and is bent by them. The forces may act at right angles to the axis of the beam, or they may be inclined at any angle. When the forces are inclined they may be resolved at the axis of the beam into rectangular components normal and parallel to the axis. The components parallel, *i.e.*, in line with the axis produce direct tension or compression in the beam; while the normal or transverse components produce the beam action or bending. The effect of the transverse forces is treated under the headings of shear and bending moment.

54. The shear at any normal section of a beam is the algebraic sum of all the *transverse* forces acting on one side (either side) of the section. Since the algebraic sum of all the transverse forces acting upon a beam in equilibrium is zero ($\Sigma V = 0$); it is evident that the shears on the two sides of a section have equal magnitudes but are opposite in sense. The shear is called positive, when acting upward on the left (of the section) or downward on the right; negative, when acting downward on the left or upward on the right.

55. The bending moment at any normal section of a beam is the algebraic sum of the moments of all the forces acting on one side (either side) of the section; taken about the center of gravity of the section as an axis. Since the algebraic sum of the moments about any point of all the forces acting upon a beam in equilibrium is zero ($\Sigma M = 0$); it is evident that the bending moments on the two sides of a section have equal magnitudes

but are opposite in rotation—the one being clockwise and the other counter-clockwise. The bending moment is called positive when clockwise on the left of and about the section, or counter-clockwise on the right of and about the section; negative, when counter-clockwise on the left of and about the section, or clockwise on the right of and about the section.

56. The Bending Moment Determined from the Location-direction Diagram.—In general the shear and bending moment vary from section to section along a beam. They may be represented conveniently for inspection by diagrams, the ordinates of which represent the shear or bending moment at various sections of the beam. Each diagram has a base or datum line. Positive values are laid off above, and negative values below this line. These diagrams have a definite relation to the magnitude-direction and location-direction diagrams, which are drawn in the graphic solution of a system of parallel forces. The diagrams in Figs. *22a* and *22b* were drawn to determine the magnitude of Q and the magnitude and direction of P. These diagrams are reproduced with some additions in Fig. 55. The shear at any section may be found from the magnitude-direction diagram. The shear on any section between the left reaction and the first load is $fa = +5$ lb. The shear on any section between the first and second loads is $fb = +3$ lb. The shear on any section between the second and third loads is $fc = -2$ lb., etc.

The bending moment at any section x between the first and second loads is

$$M_x = pP - sS$$

Produce the location-directions *ob* and *of*, to intersect at I. The point I locates the resultant of the two forces P and S; and the magnitude of this resultant is the magnitude-direction *fb*, therefore

$$pP - sS = \overline{fb}.r$$

The triangles GIJ and *bof* are similar, hence

$$y : r :: \overline{fb} : h$$

then $$\overline{fb}.r = yh$$

or $$M_x = yh$$

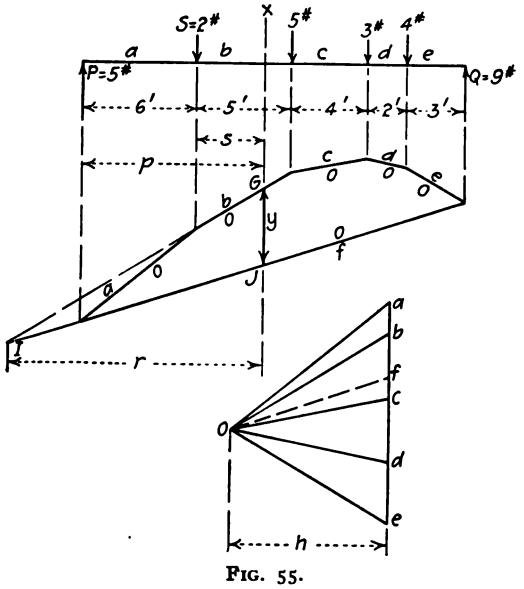
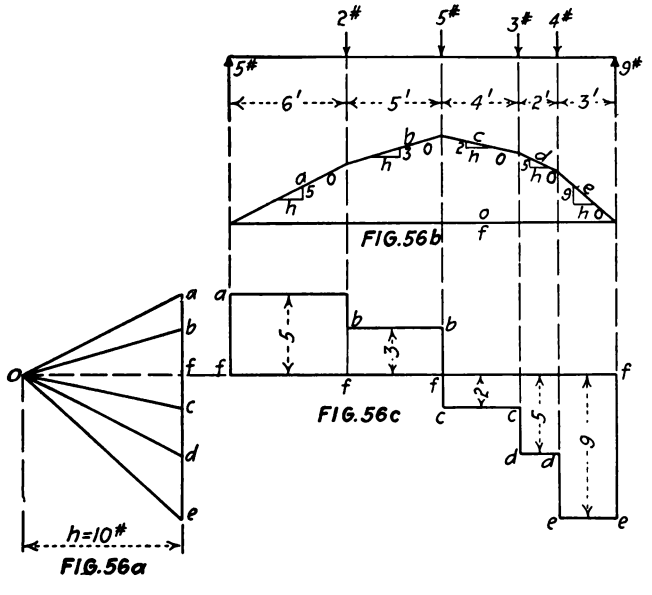

S=2#
5#
3#
4#
a
b
c
d
e
P=5#
Q=9#
6'
5'
4'
2'
3'
s
p
G
y
J
r
h

FIG. 55.


2#
5#
3#
4#
5#
9#
6'
5'
4'
2'
3'
FIG.56b
FIG.56c
h=10#
FIG.56a

Hence, the bending moment at any section equals the corresponding ordinate y in the location-direction diagram, multiplied by the horizontal component of the forces in the magnitude-direction diagram. The ordinates y are measured in feet, and the component h is measured in pounds; hence, the bending moment is expressed in foot-pounds.

If the reactions had been determined algebraically, the magnitude-direction diagram could have been drawn and closed; locating the point f, before the location-direction diagram was drawn. It would then have been possible to have chosen a point on a horizontal line through f for the pole o, (Fig. 56*a*). In this case the location-direction *of* becomes a horizontal line as shown in Fig. 56*b*; and the bending moment at any section equals 10 lb. times the number of feet scaled on the corresponding ordinate in the location-direction diagram. It has been shown that the shear at any section may be obtained from the magnitude-direction diagram. The shear at various sections may be represented more clearly by the *shear diagram* (Fig. 56*c*). The datum or base line is *ff*, and the shear at any section is represented by the ordinate at that section.

It is now clearly seen that the slope or bevel of each line in the location-direction diagram is *proportional to the shear;* for if $h = 10$ lb. is taken as the horizontal side of each bevel, the vertical side of the bevel will represent the shear. Having established the principle that the bending moment may be derived from the location-direction diagram, and that the slope of any side of the location-direction diagram is a function of the shear; we shall proceed to consider the topic of shear and moment diagrams from another point of view.

57. Shear and Bending Moment Diagrams.—The graphic method outlined in the previous article, whereby the shear and bending moment may be obtained from the magnitude-direction and location-direction diagrams, is not the one most commonly used by practicing engineers. This is especially true when the beam under consideration supports a uniform load over the whole or a portion of its length. A more convenient method, which is semi-graphical, will now be presented.

58. Illustrative Problems.

1. Draw the shear and bending moment diagrams for the beam shown in Fig. 57.

The shear diagram is drawn first because it is the simpler of the two. Draw a vertical line through the location of each load and reaction, and let ST be the base line for the shearing forces. The shear at any section between the left reaction and the first load is $+5$ lb. Lay off $SA = +5$ lb. to any convenient scale and draw AB parallel to ST. On the ordinate through B, lay off downward $BC = -2$ lb. and draw CD. Lay off $DF =$

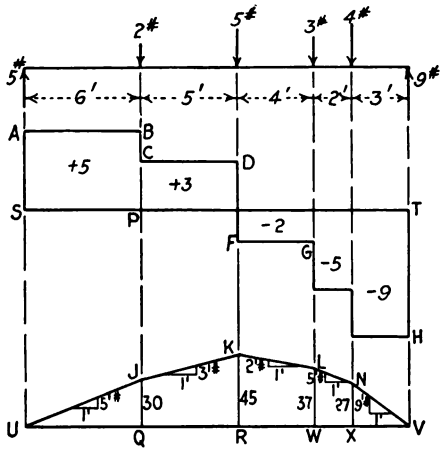


FIG. 57.

-5 lb. and draw FG. Continue in like manner to H on the ordinate through the right support, and note that HT scales 9 lb., which equals the right reaction. The broken line $SABC \ldots HT$ is the shear diagram, since the measure of any ordinate gives the shear at the corresponding section of the beam.

For practical purposes it is seldom necessary to draw the shear diagram to scale. A sketch will usually serve equally as well.

We now proceed to sketch the bending moment diagram on the base line UV. The bending moment, in accordance with

its definition, is zero at each end of the beam; consequently the diagram may be said to begin at U and end at V. In Article 56 it was shown that the slope of the bending moment diagram was proportional to the shear. The shear from A to B is positive, hence the line UJ is sketched having a positive slope (*i.e.*, upward to the right). The shear from C to D is also positive, but not as great as the shear from A to B; hence JK is sketched having a positive but lesser slope than the line UJ. The shear from F to G is negative and the line KL is sketched with a negative slope (*i.e.*, downward to the right). The slopes of LN and NV are negative, NV having the steeper slope. Thus a general idea of the variation in bending moment may be obtained from a simple sketch.

The horizontal unit of measure in both diagrams is 1 ft. The vertical unit of measure is 1 lb. in the shear diagram, and 1 ft.-lb. in the bending moment diagram. Let 1 ft. be taken as the length of the horizontal side of the triangle, representing the slope or bevel of each line in the bending moment diagram; then the length of the vertical side of each triangle will be measured in foot-pounds, and the ratio of the vertical side to the horizontal side will represent pounds and equal the shear. Thus the slope of the line UJ is $\frac{5 \text{ ft.-lb.}}{1 \text{ ft.}} = 5$ lb.; the slope of the line KL is $-\frac{2 \text{ ft.-lb.}}{1 \text{ ft.}} = -2$ lb., etc. The length of the critical ordinates JQ, KR, etc., may be obtained from the similarity of triangles.

$$UQ : 1' :: QJ : 5'\#$$

or

$$6' : 1' :: QJ : 5'\#$$

whence

$$QJ = \frac{6' \times 5'\#}{1'} = 6' \times 5\# = 30'\# = \text{area } SABP$$

From this it is clear that the difference in lengths of any two ordinates in the bending moment diagram equals the area of the shear diagram between the ordinates. The lengths of the consecutive ordinates may be computed as follows:

$$
\begin{aligned}
&\qquad\qquad 0 = \text{ordinate at } U \\
+5 \times 6 = &\ \underline{+30} \\
&\qquad +30 = JQ \\
+3 \times 5 = &\ \underline{+15} \\
&\qquad +45 = KR \\
-2 \times 4 = &\ \underline{-8} \\
&\qquad +37 = LW \\
-5 \times 2 = &\ \underline{-10} \\
&\qquad +27 = NX \\
-9 \times 3 = &\ \underline{-27} \\
&\qquad\qquad 0 = \text{ordinate at } V.
\end{aligned}
$$

This method of calculation checks itself, and is particularly helpful when the shear and bending moment diagrams are

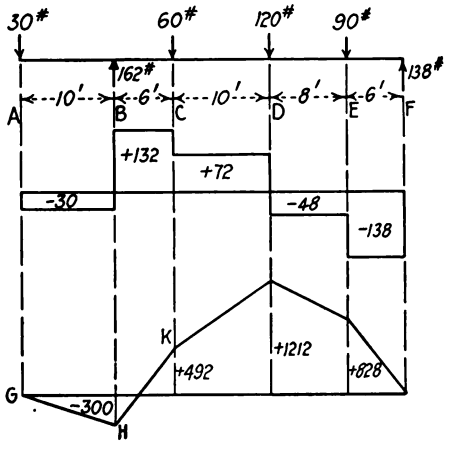


FIG. 58.

simply sketched and not drawn to scale. After the critical ordinates have been computed, the bending moment diagrams may be drawn to scale.

In going from left to right along the beam, it is obvious that the bending moment is increasing as long as the shear is positive and the lines in the bending moment diagram continue to have a positive slope. At the section where the shear changes from positive to negative, the slope changes from positive to negative and the bending moment is a maximum.

2. The shear and bending moment diagrams for a cantilever beam are sketched in Fig. 58. Between A and B the shear is negative and GH has a negative slope. At B the shear changes from negative to positive, hence HK has a positive slope, and a maximum negative bending moment occurs at B. At D the shear changes from positive to negative, giving a maximum positive bending moment. The computations for the bending moments at critical ordinates follow:

$$\begin{aligned} &0 \text{ at } A \\ -30 \times 10 &= -300 \\ &-300 \text{ at } B \\ +132 \times 6 &= +792 \\ &+492 \text{ at } C \\ +72 \times 10 &= +720 \\ &+1{,}212 \text{ at } D \\ -48 \times 8 &= -384 \\ &+828 \text{ at } E \\ -136 \times 6 &= -828 \\ &0 \text{ at } F \end{aligned}$$

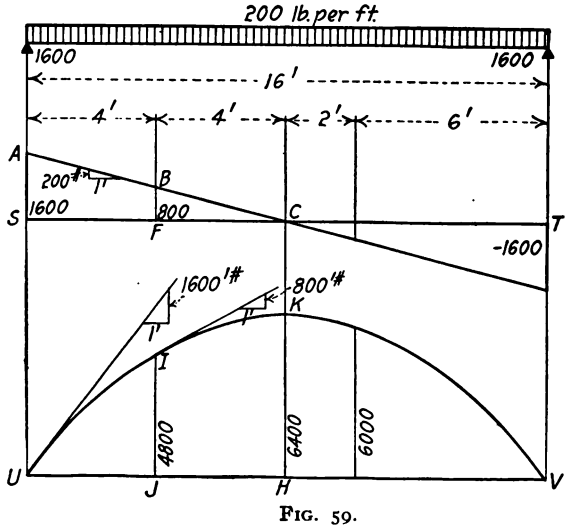


FIG. 59.

3. The beam in Fig. 59 supports a uniform load of 200 lb.

per foot. The shear diagram starts with a positive ordinate = 1,600 lb. at S, decreasing uniformly 200 lb. in each foot from S to C. Beyond C the ordinates are negative, and increasing 200 lb. per foot to −1,600 lb. at T. Since the ordinates in the shear diagram are uniformly varying, the slope of the moment diagram will be uniformly varying; and the bending moment diagram will be represented by a curved line instead of a series of broken lines. The ordinates in the shear diagram are positive

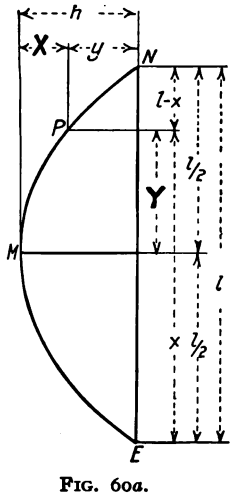


FIG. 60a.

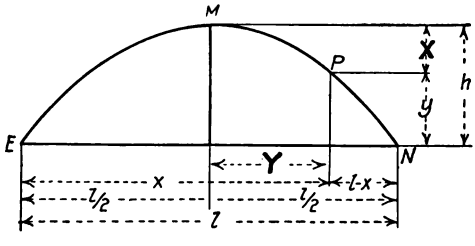


FIG. 60b.

over the left-half of the beam, decreasing to zero at the center; hence the bending moment curve has a positive slope at U, which decreases to zero at K, where the tangent to the curve is horizontal. Beyond K the curve has an increasing negative slope.

Any ordinate in the bending moment diagram is easily determined from the area of the shear diagram; thus

$$IJ = \text{area } SABF = 4{,}800 \text{ ft.-lb.}$$
$$KH = \text{area } SAC = 6{,}400 \text{ ft.-lb.}$$

The bending-moment curve is a parabola, as will be shown in the following article.

59. The Structural Engineer's Parabola.—The mathematician usually derives the equation of the parabola with the

origin at the vertex and the axis of symmetry horizontal, as shown in Fig. 60*a*. Under these conditions the general equation of the parabola is

$$Y^2 = kX \tag{1}$$

where X and Y are variables and k is a constant.

The structural engineer finds it more convenient to choose some point *not* at the vertex for the origin, with the axis of symmetry vertical as shown in Fig. 60*b*. Let P be any point on the curve in either Fig. 60*a* or *b*. Let X and Y represent the coordinates of P when the origin is at the vertex M, and let x and y represent the coordinates when the origin is at some other point E. When the point P is at N

$$X = h, \text{ and } Y = \frac{l}{2}$$

Substituting these values in Eq. (1)

$$\frac{l^2}{4} = kh$$

or

$$k = \frac{l^2}{4h}$$

Hence the mathematician's equation of this particular parabola is

$$Y^2 = \frac{l^2}{4h} X \tag{2}$$

Substituting $X = h - y$, and $Y = x - \frac{l}{2}$

in Eq. (2)

$$\left(x - \frac{l}{2}\right)^2 = \frac{l^2}{4h}(h - y)$$

or

$$y = \frac{4h}{l^2} x (l - x) \tag{3}$$

Equation (3) may be called the engineer's equation of the parabola, in the same sense that Eq. (2) may be said to represent the mathematician's equation of the same curve. Equation (3) may be written in the more general form

$$y = cx(l - x) \tag{4}$$

where x and y are variables and c is a constant. Equation (4) shows that one variable (y) is proportional to the product of two

other variables (x) and ($l - x$) whose sum (l) is constant. Any equation which can be expressed in this form is a parabolic equation.

The beam in Fig. 61 supports a uniform load of w pounds per unit length. The bending moment at any distance x from the left support is

$$M_x = \frac{wlx}{2} - \frac{wx^2}{2} = \frac{w}{2}\, x\,(l - x) \qquad (5)$$

This is a parabolic equation and the bending moment diagram is a parabola. The following important rule may be deduced from Eq. (5).

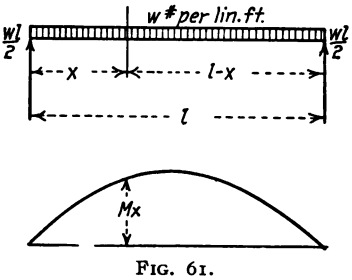


FIG. 61.

The bending moment at any section of a uniformly loaded beam equals one-half the load per unit of length, times the product of the two segments into which the section divides the span.

Thus the bending moment at JI (Fig. 59) is

$$100 \times 4 \times 12 = 4{,}800$$

Two Methods of Constructing a Parabola.—There are several methods by which a parabola may be constructed, but only two will be considered here: (*a*) an algebraic method by points; and (*b*) a graphic method by tangents.

(*a*) *Algebraic Method.*—Equation (3) may be written in the form

$$y : h :: x(l - x) : \left(\frac{l}{2}\right)\left(\frac{l}{2}\right)$$

from which the following observation may be made in connection with Fig. 59. Any two ordinates IJ and KH are to each other as the products of the two parts into which each ordinate divides the base line UV. This equation gives a clue to a very simple method of locating any desired number of points through which a parabola is to be drawn. Suppose it is desired to locate seven points on the curve. Divide the line UV into eight equal parts by seven points. The ordinate at

each point is proportional to the product of the number of parts on either side of it, thus:

$$
\begin{aligned}
1 \times 7 &= 7 \\
2 \times 6 &= 12 \\
3 \times 5 &= 15 \\
4 \times 4 &= 16 \\
5 \times 3 &= 15 \\
6 \times 2 &= 12 \\
7 \times 1 &= 7
\end{aligned}
$$

Now if we wish the middle ordinate to equal 6,400 instead of 16, we multiply that ordinate and all the others by 400, whence

$$
\begin{aligned}
7 \times 400 &= 2{,}800 \\
12 \times 400 &= 4{,}800 \\
15 \times 400 &= 6{,}000 \\
16 \times 400 &= 6{,}400, \text{ etc.}
\end{aligned}
$$

(*b*) *Graphic Method.*—If the tangents at the extremities of a parabolic curve are known, the parabola may be constructed

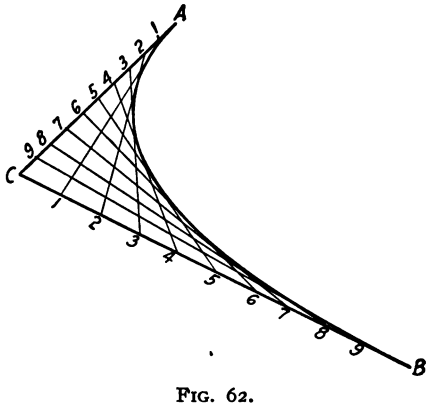


FIG. 62.

as shown in Fig. 62. Suppose we have two tangents AC and BC and the points of tangency A and B. Divide AC into any number of equal parts, and BC into the same number of equal parts. Number the points from A to C and from C to B. Connect 1-1, 2-2, etc. The parabolic curve lies tangent to these lines as shown.

Draw tangents through U and V (Fig. 59) intersecting at D. It may be observed from the shear diagram that the positive slope of the tangent UD is 1,600 ft.-lb. vertical to 1 ft. horizontal. The tangent DV has a corresponding negative slope, hence the intersection D is directly above K and the ordinate HD = 12,800 ft.-lb. Construct the parabolic bending moment diagram on these tangents, and check the lengths of several ordinates in the moment diagram by the algebraic method.

60. Illustrative Problem.

The shear and bending moment diagrams for a beam (Fig. 63) supporting a uniform load over a portion of its length, have several interesting properties. The shear diagram is drawn first, and the section at which the shear changes from positive to negative is located. The ordinates in the shear diagram indicate that the moment diagram is composed of the straight line OA, the parabolic curve $ABCD$, and the straight lines DE and EX. The maximum ordinate RC is at the section of zero shear. The following ordinates are obtained from the area of the shear diagram:

$$\begin{aligned}
&\qquad\qquad\qquad\quad 0 \text{ at } O\\
+4{,}000 \times 5 &= \underline{+20{,}000}\\
&\qquad\quad +20{,}000 = NA\\
\tfrac{1}{2}(4{,}000 + 1{,}000)15 &= \underline{+37{,}500}\\
&\qquad\quad +57{,}500 = JB\\
+1{,}000 \times \tfrac{1}{2} \times 5 &= \underline{+2{,}500}\\
&\qquad\quad +60{,}000 = RC\\
-2{,}000 \times \tfrac{1}{2} \times 10 &= \underline{-10.000}\\
&\qquad\quad +50{,}000 = WD\\
-2{,}000 \times 15 &= \underline{-30{,}000}\\
&\qquad\quad +20{,}000 = ZE\\
-5{,}000 \times 4 &= \underline{-20{,}000}\\
&\qquad\qquad\qquad\quad 0 \text{ at } X
\end{aligned}$$

Lay off to scale the ordinates NA, WD and ZE; and draw the lines OA, DE and EX. The line OA, having the same slope at A as the parabola, is tangent to it; likewise the line DE is tangent to the parabola at D. If the lines OA and ED are

produced, they will intersect at F; a point directly under the center of the uniform load. If the uniform load of 6,000 lb

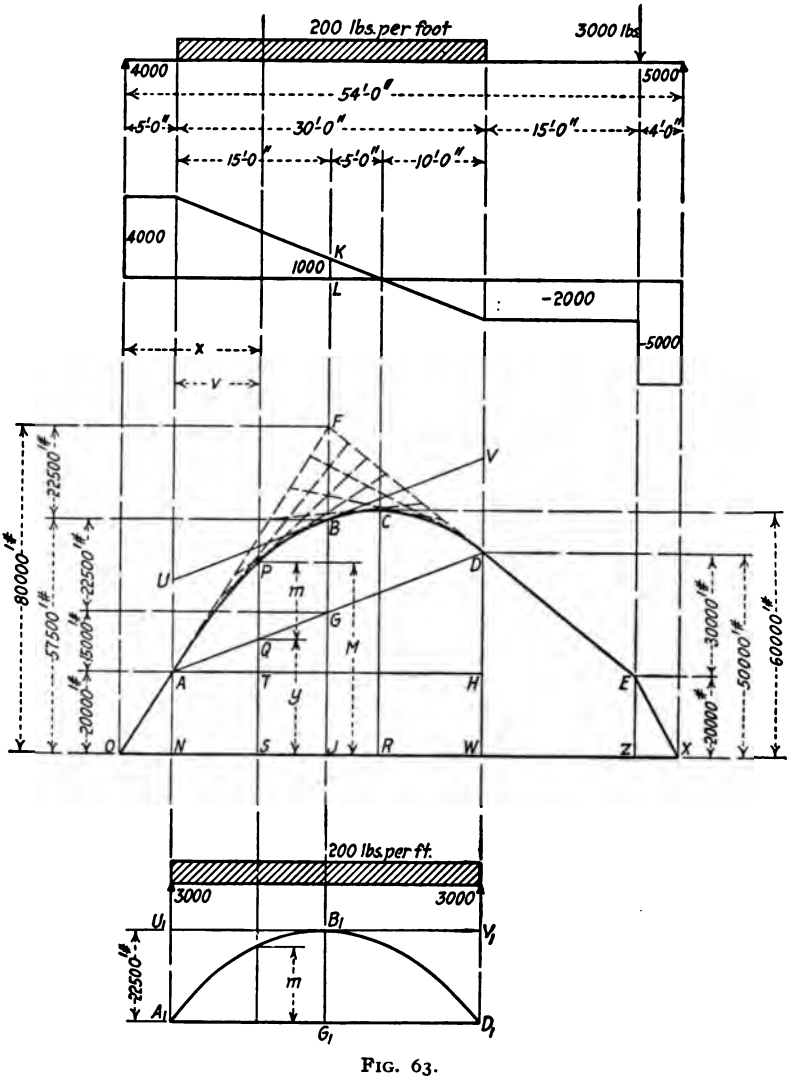


FIG. 63.

were concentrated at its center, the bending moment diagram would be $OFEX$. The parabola may now be drawn on the tangents FA and FD.

The parabolic segment $ABDG$ has several interesting properties. Let PS represent any ordinate in the bending moment diagram between AN and DW, and let x represent the distance of this ordinate from the left support; then the bending moment at x is

$$M_x = PS = 4{,}000x - \frac{200(x-5)^2}{2}$$
$$= -100x^2 + 5{,}000x - 2{,}500 \qquad (1)$$

Let $QS = y$ be the ordinate to the line AD, then from similar triangles

$$QT:DH::TA:HA$$
$$QT = y - AN = y - 20{,}000$$
$$DH = DW - AN = 30{,}000$$
$$TA = x - 5$$
$$HA = 30$$

therefore $y - 20{,}000:30{,}000::x - 5:30$

or $y = 1{,}000x + 15{,}000$

then $PQ = M_x - y = m = -100x^2 + 4{,}000x - 17{,}500$

Let $AT = x - 5 = v$

or $x = v + 5$

then $m = -100(v+5)^2 + 4{,}000(v+5) - 17{,}500$

or $m = 100v(30 - v) \qquad (2)$

Equation (2) is also the expression for the bending moment m at any section of a beam 30 ft. long, supporting a uniform load of 200 lb. per foot when the section is a distance v from either support. Therefore the lengths of corresponding ordinates in the two parabolic segments $ABDG$ and $A_1B_1D_1G_1$ are equal. The areas of the two segments are equal and the centers of gravity are similarly situated.

The slope of the line AD is $\frac{DH}{HA} = 1{,}000$ lb. The shear ordinate KL is 1,000 lb.; hence the line UV, which is tangent to the parabola at B, has a slope of 1,000 lb., and is therefore parallel to the line AD. The point B bisects the line FG.

61. Problems.

1–7. Determine the reactions algebraically, sketch the shear and bending moment diagrams, compute the values for critical ordinates and draw the diagrams to scale for the beams shown in Figs. 64 to 70 inclusive. The loads A

and B in Fig. 65 are equal. In Fig. 66, A, B and C are frictionless pegs. The rope passes over the upper pegs and under the lower one. The beams in Figs. 67, 68 and 69, are fixed in a wall at the left end.

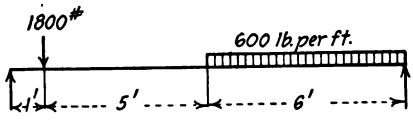


FIG. 64.

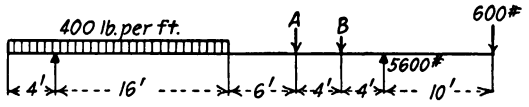


FIG. 65.

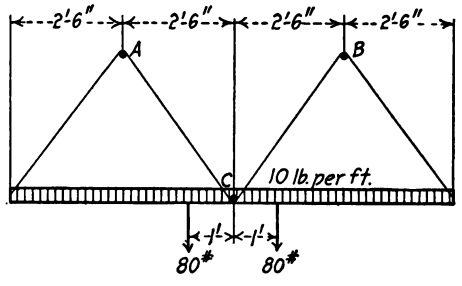
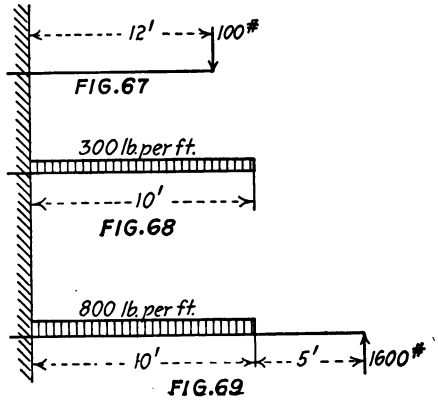


FIG. 66.



8. A timber 20 ft. long, weighing 40 lb. per linear foot is floating in water, and supports 200 lb. 4 ft. from the left end and 150 lb. 2 ft. from the right end. Draw the shear and bending-moment diagrams.

9. A timber 80 ft. long, floating in water, supports three loads of 80 lb. each, one at the center and one at each quarter point. Neglect the weight of the timber and draw shear and bending moment diagrams.

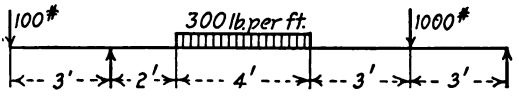


FIG. 70.

10. A beam 30 ft. long is supported at each end A and C. A uniform load of 100 lb. per foot extends from A to B, and a uniform load of 225 lb. per foot extends from B to C. The bending moment is a maximum at B. Locate B, and draw the shear and bending-moment diagrams.

SEC. IV. FRAMES HAVING MEMBERS WHICH PERFORM THE FUNCTIONS OF A BEAM

62. The framed structures heretofore discussed have supported loads applied only at the joints. This is one of the four conditions mentioned in Article **40**, which must be met if the members of a structure are subject to longitudinal stresses (tension or compression) only. In order to prepare the way for an analysis of a structure composed of a roof truss and columns, in which the columns also serve as beams when the structure is resisting inclined loads; it will be advisable to consider several types of simple frames which resist the action of loads applied not only at the joints, but also at intermediate points. Such members must be designed to resist shearing and bending stresses, as well as longitudinal stresses. The analysis of a structure of this type may be made by a process of dissection, whereby each member is sketched separately and the forces acting thereon are shown.

63. Illustrative Problem.

The frame shown in Fig. 71, hinged at A and resting on a roller at B, supports a load of 300 lb. at E. We find first the reactions or the forces necessary for the equilibrium of the frame as a whole. On account of the roller at B, the reaction there is horizontal; consequently there is a vertical force of 300 lb. acting upward at A, and a horizontal force equal and opposite to the horizontal force at B. Balance the moments about A.

$$\frac{300 \times 6}{24} = 75$$

46536

The horizontal force of 75 lb. acts to the left at B, and to the right at A. Now sketch each member separately.

The vertical member resists the action of forces at four points, A, B, C and D. The forces at A and B are known. The unknown forces at C and D are the result of the interaction of the members, and are represented by horizontal and vertical components. The horizontal member resists the action of forces at three points, the force at E being known. Indicate horizontal and vertical components at D' and F. The inclined member resists forces at two points F' and C', as shown. Bal-

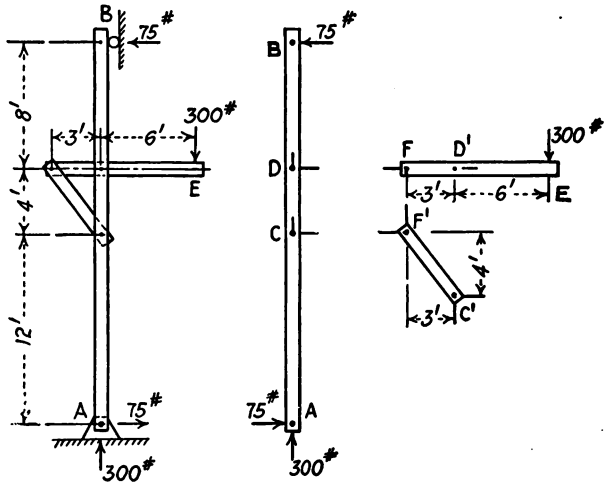


FIG. 71.

ance the moments of the forces acting on the horizontal member about D'; the vertical force at F is 600 lb. acting downward. *Indicate it at once on the sketch; and do likewise with each force, as soon as its magnitude and sense are found.* The vertical force at D', is 900 lb. acting upward. Since the inclined member is pulling downward 600 lb. at F, the horizontal member must be pulling upward 600 lb. at F'. A vertical force of 600 lb. acts downward at C', and upward at C. A vertical force of 900 lb. acts downward at D. Balance the moments of the forces acting on the inclined member about F'.

$$\frac{600 \times 3}{4} = 450$$

A horizontal force of 450 lb. acts to the right at C', and to the left at F'; consequently a horizontal force of 450 lb. acts to the left at C and to the right at F. Finally a horizontal force of 450 lb. acts to the left at D', and to the right at D.

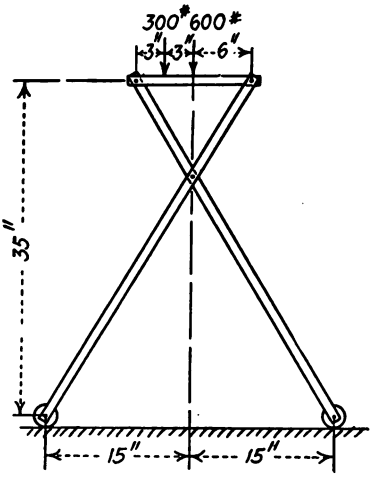


FIG. 72.

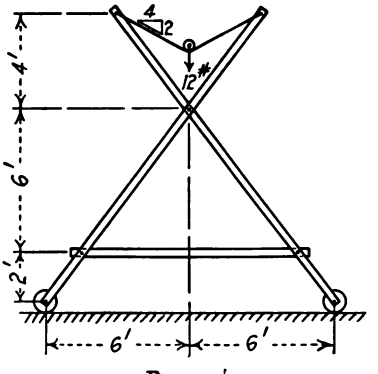


FIG. 73.

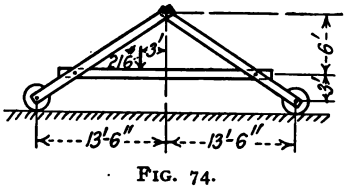


FIG. 74.

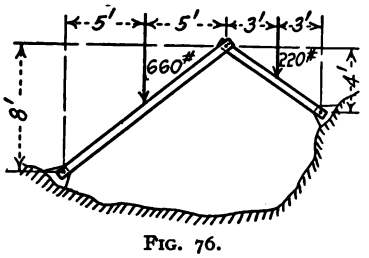


FIG. 76.

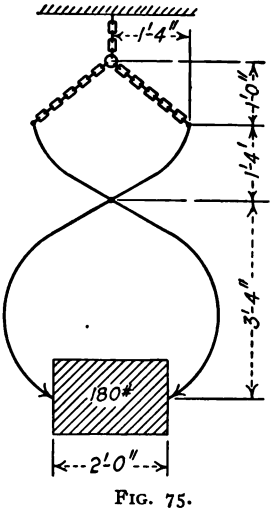


FIG. 75.

Inspect each member and see if the forces acting on each are in equilibrium. Draw shear and bending moment diagrams for the vertical and horizontal members.

The resultant of the two forces acting at F' must have a direction through C', otherwise the inclined member would rotate and not be in equilibrium. Likewise the resultant at C' passes through F', and the member resists a longitudinal stress. Whenever forces act at only two points on a member, the member reacts as a tie or strut and not as a beam.

64. Problems.

1–6. The structures shown in Figs. 72 to 77 inclusive are to be treated as follows: Determine the horizontal and vertical components of the reactions, sketch each member separately and find the horizontal and vertical components of the forces acting thereon.

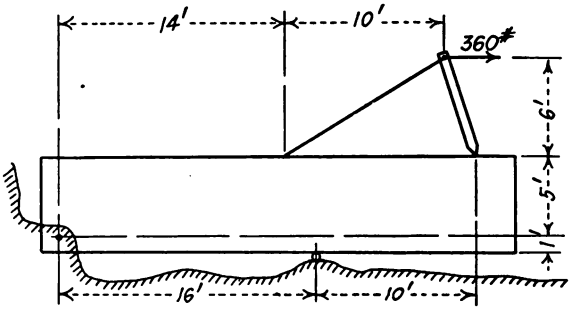


FIG. 77.

65. The portal frame (Fig. 78*a*), resisting a horizontal force of 600 lb. at D has four unknown magnitudes represented in the reactions—*i.e.* a vertical and a horizontal force at each support; but the frame, being composed of two rigid bodies hinged at C, is statically determinate. Each vertical member or column is continuous. The two horizontal members and the inclined members are hinged at C. Balance the moments of all the forces about A.

$$\frac{600 \times 24}{16} = 900$$

A vertical force of 900 lb. acts upward at B, consequently a vertical force of 900 lb. acts downward at A. Balance the moments of the forces acting on the right portion of the structure about C.

$$\frac{900 \times 8}{30} = 240$$

A horizontal force of 240 lb. acts to the left at B. Balance the moments of the forces acting on the left portion of the structure about C.

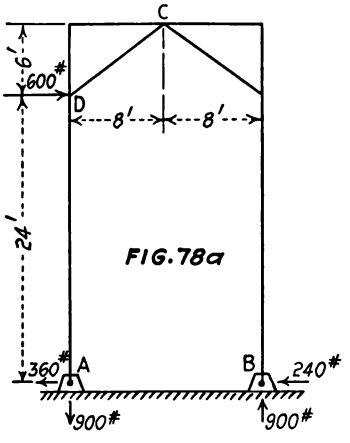


FIG. 78a

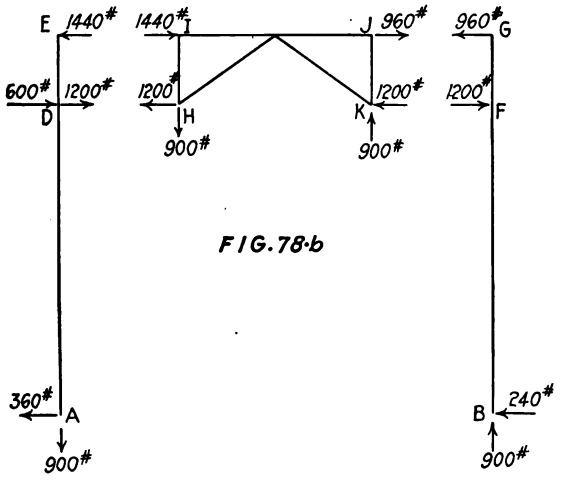


FIG. 78b

$$900 \times 8 = 7{,}200$$
$$600 \times 6 = 3{,}600$$
$$30)\overline{10{,}800}$$
$$360$$

A horizontal force of 360 lb. acts to the left at A.

Each member in the structure might now be sketched separately, and the stresses analyzed by the algebraic method, as in

Article **63.** In many cases, however, and especially if the frame is more complex, a graphic solution by means of the stress diagram is desirable; but before a stress diagram can be drawn, the bending effect of the columns upon the frame, caused by the horizontal forces at A and B, must be considered. Each column is sketched separately as before (Fig. 78*b*.) The known forces acting on the columns are the reactions at A and B and the load of 600 lb. at D. Other forces, representing the action of the frame $HIJK$ on the columns, must be supplied at E, D, F and G. It is necessary to consider only the horizontal forces required for the equilibrium of the columns at these points, since the vertical forces in the columns do not cause bending in the columns. The horizontal forces necessary for the equilibrium of the columns are as follows:

At G $\frac{240 \times 24}{6} = 960$ acting to the left.

At F $240 + 960 = 1{,}200$ acting to the right.

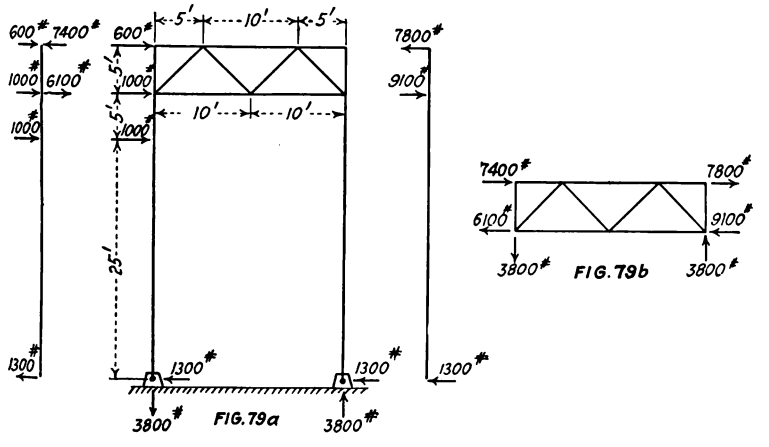
At E $\frac{360 \times 24}{6} = 1{,}440$ acting to the left.

At D $1{,}440 + 360 - 600 = 1{,}200$ acting to the right.

These four horizontal forces represent the bending action of the frame on the columns. The reaction of the columns on the frame are equal and opposite as indicated on the frame. The vertical forces acting on the columns at A and B are transferred to H and K. The frame is in equilibrium, and a stress diagram may be drawn.

66. The portal frame (Fig. 79*a*) is statically indeterminate. The vertical reaction at either support may be determined by balancing the moments of all the forces about the other support, thereby eliminating the two horizontal reactions. The horizontal reactions cannot be found by the principles of statics. In order that an approximate estimate of the stresses may be made, current engineering practice *assumes* that the horizontal load is supported equally by the two columns. The two columns are sketched separately, as before. The horizontal loads and reactions are shown on the outside of the columns and the forces, interacting between columns and frame are shown on the inside

of the columns. The latter forces have been transferred to the frame in Fig. 79*b*, and the vertical reactions indicated. A stress diagram may now be drawn.



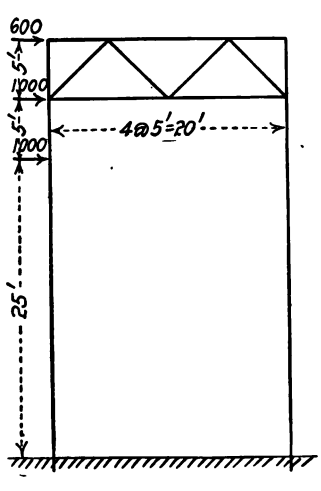


FIG. 80*a*.

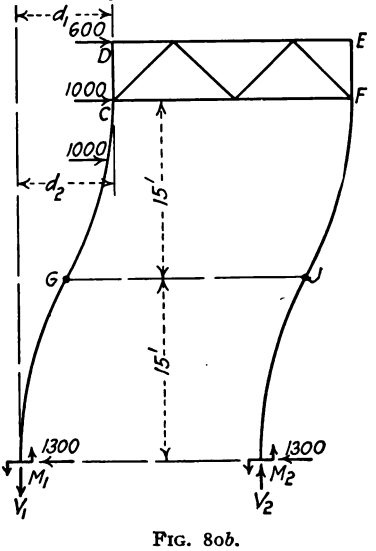


FIG. 80*b*.

67. The portal frame (Fig. 80*a*) is supported by columns which are fixed at their bases instead of hinged, and the columns are not free to rotate about their points of support. There are three unknown quantities at each support—a hori-

zontal force, a vertical force and a couple; for which six equations or statements are necessary for a solution. The principles of statics furnish three of these equations, which are to be augmented by three statements from which the three additional equations are obtained. The first supplementary equation is deduced from the same assumption that was made in Article 66, *viz.*, that each horizontal reaction is $\frac{1}{2} \times 2{,}600 = 1{,}300$ lb.

An exaggerated idea of the distortion which the structure undergoes when loaded, is illustrated in Fig. 80*b*. If the columns are rigidly fixed at their bases, a tangent to the axis of each column remains vertical after the column is bent. The difference in the displacements d_1 and d_2 is assumed negligible, when compared with either d_1 or d_2. A similar assumption applies to the points E and F; and the frame $CDEF$, although displaced from its original position, is *assumed* to remain a rectangle. On the basis of this assumption the point of contraflexure, or zero bending moment, in each column is midway between the base of the column and the bottom chord of the frame. This is another way of saying that the columns may be considered hinged at G and J.

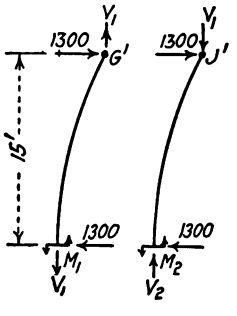


FIG. 80*c*.

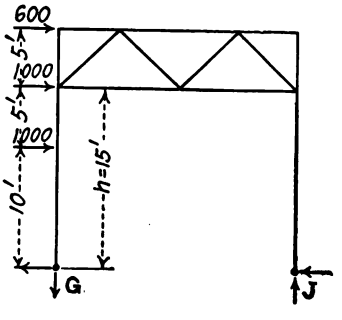


FIG. 80*d*.

The lower portions of the columns are shown in Fig. 80*c*. The horizontal displacements of G' and J' are grossly exaggerated in the figure. The couple formed by the two vertical forces is so very small when compared to the horizontal couple, that it may be neglected without appreciable error. Balance the moments of all the forces about G'.

$M_1 = 15 \times 1{,}300 = 19{,}500$ ft.-lb. counter-clockwise.

A horizontal force of 1,300 lb. acting to the right, and a vertical force V_1 acting upward must be supplied at G' by the upper part of the column, in order that the lower part may be in equilibrium. The forces required at J' (Fig. 80*c*) are likewise indicated.

$M_2 = 15 \times 1{,}300 = 19{,}500$ ft.-lb. counter-clockwise.

The forces at G' and J' (Fig. 80*c*) are shown reversed at G and J (Fig. 80*d*) and the solution may proceed as in Article **66**.

Structures of this character are seldom, if ever, designed with pinned connections at the column bases, as illustrated in Fig. 79*a*. It is equally true that the column connections to the foundation are seldom, if ever, made with sufficient rigidity to justify the assumption of zero bending moment at one-half the distance from the foot of the column to the frame, as shown in Fig. 80*b*. The point of zero bending moment in nearly all cases lies somewhere between these two extremes, and is generally assumed to be one-third the distance from the foot of the column to the first connection with the frame. This assumption makes $h = 20$ ft. in Fig. 80*d*.

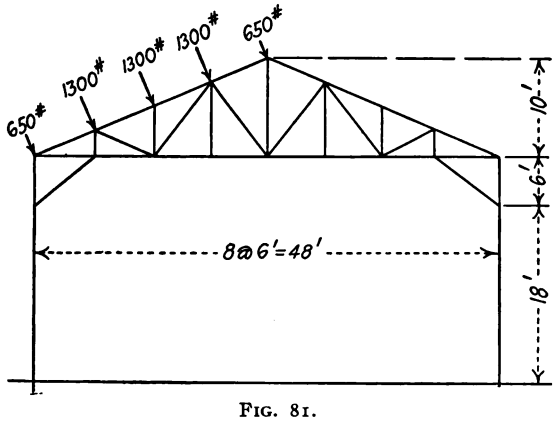


FIG. 81.

68. Problems.

1. Complete the solution for Fig. 79*b* and the two solutions in Fig. 80*d*, where $h = 15$ ft. in one instance and $h = 20$ ft. in the other. Compare the results of the three solutions.

2. Draw a stress diagram for the Fink truss in Fig. 81, assuming that the point of contraflexure in each column is one-third the distance from the base of the column to the foot of the knee brace.

CHAPTER III

ROOF TRUSSES

SEC. I. TYPES AND LOADS

69. Standard Types.—Roof trusses have been built in a great variety of forms in conformance with architectural requirements, but under ordinary conditions the form of the truss is determined by the length of span, slope of roof and the clear head room underneath. The most common types used in current practice are shown in Fig. 82.

The Howe truss is especially adapted to wood construction. The top chord and diagonals are wood, the vertical tension members are steel rods and the bottom chord is either wood or steel. This type is also used when the truss is made entirely of steel.

The Fink truss is the most common form in use for supporting roofs on which the character of the covering makes a steep slope desirable, as in the following cases: "Bonanza" tile, 5 in. per foot; corrugated sheet steel, 6 in. per foot; slate, 7 in. per foot. The advantages of this type are the short compression members, which make for economy; and the variations in number and length of panels in the top chord for any given length. When the short member in a Fink truss is replaced by two members, the structure is sometimes called a fan truss.

The Warren truss is used extensively for tar and gravel roofs with the following slopes: ¾ in. per foot for spans below 40 ft. in length, 1¼ in. per foot for spans 60 ft. and over, and 1 in. per foot for intermediate spans.

Saw-tooth trusses have top chords of unequal length. Windows are placed in the plane of the shorter legs, which face north in the northern hemisphere to avoid the direct rays of the sun.

Three-hinged arches are more economical than Fink or Warren trusses for spans above about 125 ft. in length.

Monitor trusses are auxiliary frames which may be added to the main trusses to provide light or ventilation.

A bent consists of a roof truss and the columns which support it.

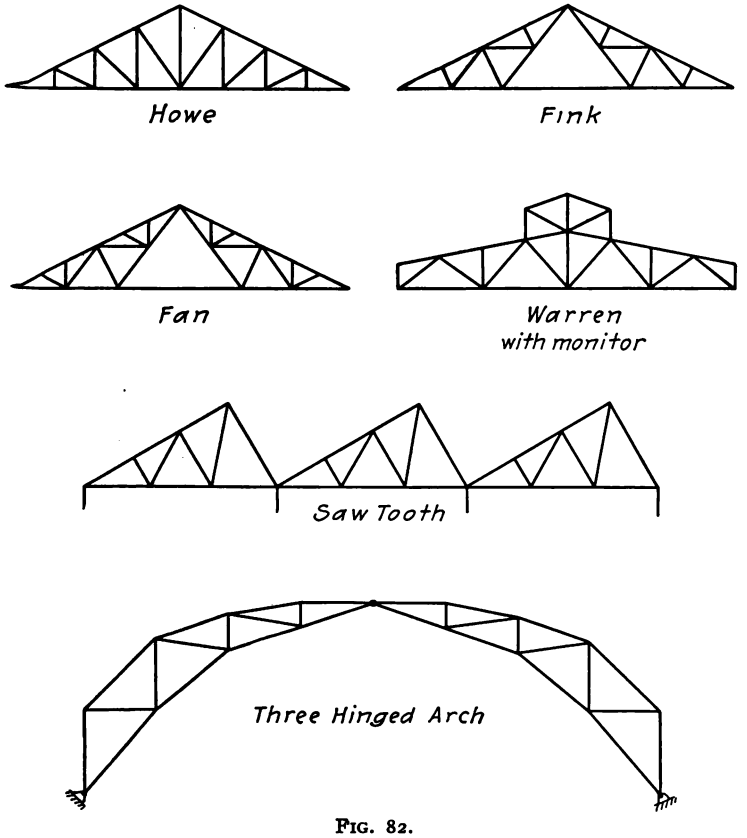


FIG. 82.

A bay is the space between adjacent trusses, and the distance center-to-center of trusses is called the bay length.

70. Purlins.—The roof covering is supported by purlins which are carried by the trusses. The purlins are usually channels, although I-beams are sometimes used. It was formerly the custom to lay out the design so that the purlins would rest on the top chord of the truss at the panel points; but current practice seems to disregard this feature by fre-

quently making the panel lengths longer than the purlin spacing. This arrangement reduces the number of web members otherwise required; but increases the size of the top chord, which is longer between panel points and must also perform the additional duty of a beam in supporting the purlins between panel points. The extra weight in the top chord more than offsets the weight saved in the web members, and thus adds weight to the structure; but the cost is less on account of time saved in fabrication.

71. The bracing is an important part of any structure and is too frequently given little attention. It plays a leading role during erection, especially if the trusses are long or the building is high; but the chief purpose of bracing is to carry the wind loads and vibratory effects of machinery and the like to the foundations. To provide for wind pressure on the ends of a building, bracing is placed in the planes of the top and bottom chords of the trusses, which transfers the wind load to the eaves. If the trusses are supported on columns, the load is transferred from the eaves to the ground by bracing between the side columns. This bracing is placed in the end bays and in approximately every fourth bay.

The wind load on the sides of a building supported on columns may be taken to the foundation by two different paths. Each truss and the columns supporting it may act as a unit, in which case the wind load on each adjacent half bay is carried to the ground through the columns acting also as a beam. This device is used whenever possible, for it is a sound economic principle to transfer a load to the ground over the shortest path possible. Whenever the building contains a crane runway, the use of knee braces from the bottom chord to columns is often objectionable or impossible. If sufficiently rigid connections between the trusses and columns cannot be made, the wind and transverse thrust from the crane may be taken to the ends of the building by diagonal bracing in every bay in the plane of the bottom chords of the trusses. At the ends of the building the load is transferred to the ground by bracing between the end columns. This method of bracing is not effective in a building so long that expansion joints are necessary.

The uncertainty concerning the nature and amount of wind load (as will be shown later) and the vibratory action of machinery; and the fact that these loads may often be carried to the ground over more than one path, makes the problem of estimating the probable resulting stresses in the bracing a very difficult one. As a matter of fact it is seldom attempted except in a structure of extraordinary capacity. The design of bracing is governed more often by the lengths of its members than by the stresses they are supposed to carry.

72. Spacing of Trusses.—The economic bay length is a function of several variables—the cost of the purlins, trusses, columns and foundations, all enter into the problem. If the bay length is from 8 to 12 ft., 2 or 3 in. tongue and grooved planking, resting directly on the trusses may be used, in which case purlins are not required. When purlins are used, the bay length is usually about 16 ft. for spans up to 65 ft. For longer spans the bay length is about one-quarter of the span.

73. The dead loads supported by a truss include the roof covering, the purlins, the bracing, the weight of the truss itself and any stationary loads, such as a machinery floor or ceiling which may be suspended from the truss.

Roof Covering.—The character and weight of the roof covering will influence the size and spacing of the purlins.

The weights per square foot of roof surface for the more common materials are as follows: shingles, 2 to 3 lb.; slate, 8 to 10 lb.; corrugated steel, 1 to 3 lb.; tile, 8 to 25 lb.; felt and gravel, 8 to 10 lb.; boards 1 in. thick, 3 to 4 lb.

The purlins support the roof covering, the live load (wind or snow or both), and their own weight. Purlins are assumed to weigh about 3 lb. per square foot of the roof surface. This assumption should be compared with the actual weight after the purlins have been designed. If the assumed and actual weights differ materially, a correction should be made, and the design modified if necessary.

Bracing.—The weight of the bracing is a relatively small item and is generally not considered, in estimating the total dead load. Whenever considered, it is usually estimated at about 1 lb. per square foot of horizontal surface.

Trusses.—The actual weight of a truss cannot be determined until the design and shop drawings have been made. It is necessary, however, to have an approximate weight of the truss when estimating the total dead load. The weight of a steel truss is a function of several variable quantities, *i.e.* the length of span, the length of bay, the load to be supported, the specifications used in the design, the type of truss, the slope of the roof, the position of the purlins with reference to the panel points, and the amount of unavoidable excess metal necessary in order that the design may be in accordance with the specifications. Empirical formulas, frequently used for finding the weights of trusses, are as a rule not trustworthy; for they include only a few of the determining factors, and the limits within which these factors are applicable are not specified. The approximate weights given in the following table may be used for the preliminary design. After the stresses have been found and the design made, a closer approximation to the actual weight may be made in the following manner. Find the weight of the members in the truss, taking the distance center to center of panel points for the length of each member; and add 25 per cent for details. If the approximate weight of the truss thus found differs from the assumed weight by an amount sufficient to change the total load (dead and live) 5 per cent, the designer should consider the desirability of revising the design.

TABLE I.—WEIGHTS OF ROOF TRUSSES IN POUNDS

Span, feet	Total load in pounds per linear foot of span					
	500	600	700	800	900	1,000
30	850	950	1,000	1,100	1,150	1,350
40	1,250	1,500	1,650	1,800	1,950	2,100
50	2,000	2,200	2,350	2,650	3,000	3,300
60	2,350	2,800	3,200	3,500	3,800	3,900
70	3,300	3,850	4,100	4,600	4,950	5,350
80	4,200	4,850	5,350	6,000	6,250	6,900

74. The snow load varies with the latitude, humidity and slope of roof. The weight of freshly fallen snow varies from 5

to 12 lb. per cubic foot, according to its dampness. The weight when packed may vary from 15 to 40 lb. per cubic foot. Snow is seldom considered, if the roof has an inclination of 45° or more to the horizontal. Specifications usually allow the following weights per horizontal square foot for snow: New England, 30 lb.; New York and Chicago, 20 lb.; Baltimore, Cincinnati and St. Louis, 10 lb.

75. The Wind Load.[1]—The effect of wind upon structures is at present an unsolved problem. Nearly all scientists since Newton's time agree that wind pressures of equal densities vary as the square of the velocity. This law may be expressed by the empirical equation

$$P = kV^2,$$

where P represents the pressure in pounds per square foot on a surface normal to the wind, V is the wind velocity in miles per hour and k is a constant factor to be determined by experiment. Scientists do not agree on the value of k, but current practice seems to consider 0.004 as a proper value; whence

$$P = 0.004V^2.$$

If this formula is accepted, the following wind pressures on a normal surface will result:

Wind Velocities Miles per Hour	Pressure, Normal Surface, Pounds per Square Foot
60	14.4
70	19.6
80	25.6
90	32.4
100	40.0

Again authorities differ as to the maximum velocity for which provision should be made. It is considered useless to design a structure sufficiently strong to resist a tornado which may come once in a lifetime and be of short duration. High velocities come in gusts and act over relatively small areas. In the light of our present knowledge, the provision for a wind pressure of 20 lb. per square foot on vertical surfaces seems ample.

The action of wind upon an inclined surface is also a per-

[1] The question of wind pressure on buildings has been presented in an admirable manner by R. FLEMING, in the *Engineering News*, Jan. 28 and Feb. 4, 1915.

plexing problem to which scientists have given much attention. It is not correct to consider the effect of the wind on an inclined roof surface to be the same as upon a vertical surface; nor is it correct to resolve the horizontal force of the wind into two components, the one normal to the roof, and the other along the surface of the roof; for the effect of the normal component only, would be resisted by the roof covering.

In 1829 Duchemin, a French army officer, investigated the wind pressure on inclined surfaces; and his conclusions, approximately verified by Langley in 1888, have been generally accepted. Duchemin's empirical formula is:

$$P_n = P \frac{2 \sin A}{1 + \sin^2 A}$$

in which P_n = pressure normal to the roof,
P = pressure on a vertical surface,
A = angle of inclination to the horizontal.

When $A = 45°$ or more, P_n should be taken equal to P. The normal pressure for various slopes and angles of inclination are as follows:

TABLE II.—NORMAL WIND PRESSURES ON ROOF SURFACES

Angle of inclination or slope A	Slope	Normal pressures in pounds per square foot when pressure on vertical surface is P = 20 lb. per square foot
4° 46′	1 in. to 1 ft.	3.3
5° 0′		3.5
9° 28′	2 in. to 1 ft.	6.4
10° 0′		6.8
14° 2′	3 in. to 1 ft.	9.2
15° 0′		9.6
18° 26′	4 in. to 1 ft.	11.5
20° 0		12.2
22° 37′	5 in. to 1 ft.	13.4
25° 0′		14.4
26° 34′	6 in. to 1 ft.	14.9
30° 0′		16.0
30° 15′	7 in. to 1 ft.	16.1
33° 41′	8 in. to 1 ft.	17.0
35° 0′		17.3
40° 0′		18.2
45° 90′		20.0

SEC. II. STRESS ANALYSIS FOR WIND

76. Wall Bearing Trusses.—The stresses in a roof truss are usually determined by graphic methods. Theoretically, two stress diagrams are required—one for vertical loads, and one for the wind loads acting normal to the roof. In practice, however, a separate stress diagram for wind is seldom drawn, except for trusses supported on columns; and then only for localities having high winds and no snow. The reason for this

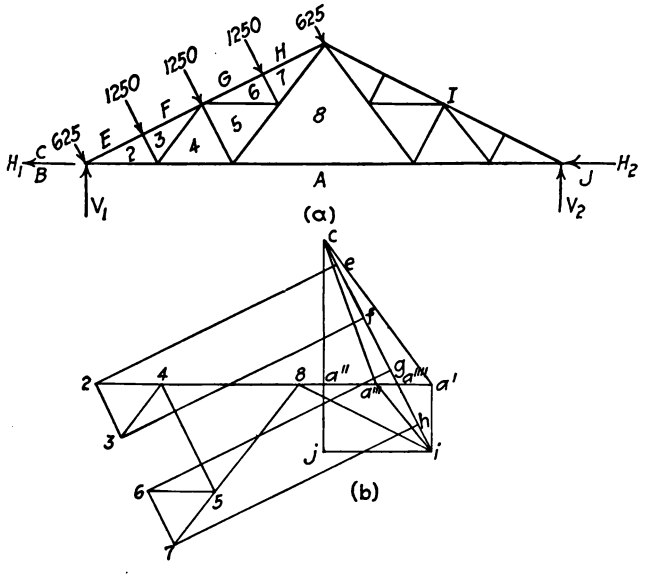


FIG. 83.

will now be shown in connection with the Fink truss in Fig. 83*a*. The following data will be used: The truss is supported on walls; span, 40 ft.; rise, 10 ft.; length of bay, 15 ft.; wind pressure, 20 lb. per square foot on a vertical surface. The normal wind pressure, taken from the table in Article **75**, is 14.9 lb. per square foot of roof surface. The length of the top chord is 22.36 ft.; hence, the area subject to wind pressure which each truss supports is $22.36 \times 15 = 334.8$ sq. ft.; and the total wind load is $334.8 \times 14.9 = 5{,}000$ lb.; or 1,250 lb. per panel. A half-panel load is supported at the eaves and peak, and a full panel load at each of the three other panel points.

The horizontal and vertical components of the total load are 2,236 and 4,472 lb. respectively. Let V_1, H_1, V_2 and H_2 represent, respectively, the vertical and horizontal components of the left and right reaction (see Article **52**).

$$V_2 = \frac{5{,}000 \times 11.18}{40} = 1{,}398$$

$$V_1 = 4{,}472 - 1{,}398 = 3{,}074$$

Since the horizontal reactions are statically indeterminate, four assumptions as to the values of H_1 and H_2 will be considered.

Case I.—The total horizontal reaction is carried by the left support.

$$H_1 = 2{,}236$$
$$H_2 = 0$$

Case II.—The total horizontal reaction is carried by the right support

$$H_1 = 0$$
$$H_2 = 2{,}236$$

Case III.—The horizontal reactions are equal

$$H_1 = H_2 = 1{,}118$$

Case IV.—The horizontal reaction at each support is proportional to the vertical reaction

$$H_1 = \frac{3{,}074}{4{,}476} \times 2{,}236 = 1{,}537$$

$$H_2 = \frac{1{,}398}{4{,}476} \times 2{,}236 = 699$$

The stresses for all four cases, determined by the stress diagram in Fig. 83*b*, are recorded in Table IV; where each member is numbered to correspond with Fig. 87. The bottom chord is straight, and in line with the two points of support; and because of this, the stress in any member, except those of the bottom chord, is the same for the four different assumptions regarding the horizontal reactions.

77. Trusses Supported on Columns.—*The Mill Building Bent.*—When the truss of Fig. 83*a* is supported by columns, as shown in Fig. 84; six unknown quantities are involved in the reactions—a vertical force, a horizontal force and a resisting

moment at each support. The problem is therefore statically indeterminate in the third degree, and the three static equations of equilibrium must be augmented by three *elastic equations*, in making an exact analysis of the reactions. The derivation of these elastic equations is a long and involved process, and will not be considered here. In place of the elastic equations, three assumptions are made. Hitherto, it has been the practice to treat the mill building bent as a portal frame, the same assumptions being made as outlined at the end of Article 67; *viz.*, that the point of contraflexure, or zero bending moment, in each column is one-third of the distance from the base of the column to the foot of the knee-brace; and the total horizontal shear is equally resisted by each column.

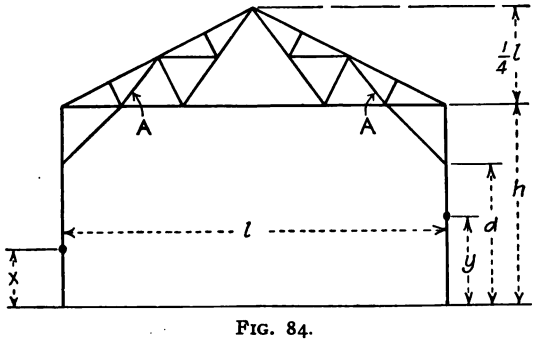


FIG. 84.

A recent investigation,[1] made by Mr. S. R. Offutt at the University of Illinois, under the direction of the author has proved that the assumption of equal horizontal reaction is no longer justified.

Twenty structures having the general outline, as shown in Fig. 84, and dimensions given in Table III were designed and investigated. Fink trusses, having a slope of 6 in. to 1 ft., were used in all cases. Each truss had eight panels, except the 20-ft. trusses, which had four panels. The length of the bay was 15 ft. The truss for each span length l was designed for a uniform vertical load of about 40 lb. per square foot of horizontal surface. This load was varied somewhat in several

[1] This investigation is to be published as a Bulletin of the University of Illinois Engineering Experiment Station.

instances to note the effect upon the final results. The members A were made about twice as large as was required for the vertical load. The knee-braces were designed for a wind pressure of 20 lb. per square foot on the vertical height h, and 14.9 lb. per square foot of roof surface normal to the roof. The columns in general were designed to support their vertical load and the additional effect of bending; by assuming that the point of contraflexure was one-third the distance from the base to the knee-brace; and that the horizontal reactions were equal.

TABLE III.—RATIOS FOR HORIZONTAL REACTIONS AND POINTS OF CONTRAFLEXURE IN MILL BUILDING COLUMNS

Span length	Column height	Ratio	Pin-connected	Rigidly fixed		
l	h	$\frac{l}{h}$	$\frac{H_R}{H}$	$\frac{H_R}{H}$	$\frac{x}{d}$	$\frac{y}{d}$
1	2	3	4	5	6	7
20	10	0.500	0.399	0.385	0.518	0.654
20	12	0.600	0.379	0.354	0.486	0.620
20	16	0.800	0.358	0.323	0.435	0.585
20	19	0.950	0.356	0.309	0.439	0.569
30	16	0.533	0.380	0.353	0.481	0.593
30	21	0.700	0.364	0.324	0.435	0.570
30	26	0.867	0.354	0.309	0.432	0.547
30	31	1.033	0.340	0.290	0.439	0.562
40	16	0.400	0.410	0.401	0.621	0.612
40	21	0.525	0.385	0.356	0.488	0.581
40	26	0.650	0.373	0.335	0.471	0.560
40	31	0.775	0.358	0.314	0.457	0.568
50	16	0.320	0.425	0.429	0.634	0.601
50	21	0.420	0.397	0.375	0.498	0.571
50	26	0.520	0.382	0.349	0.465	0.555
50	31	0.620	0.369	0.332	0.452	0.558
60	16	0.267	0.439	0.453	0.643	0.593
60	21	0.350	0.407	0.388	0.494	0.574
60	26	0.433	0.390	0.361	0.454	0.558
60	31	0.517	0.379	0.345	0.429	0.554

This column design also was varied in several instances to note the general effect. Each bent thus designed was then analyzed, strictly in accordance with the elastic theory of structures by the method of deflections. This theory is too complicated to be explained here. One assumption, not strictly in accord with actual conditions was made; *viz.*—that the truss and knee-brace members were pin-connected. The horizontal wind loads were concentrated at points about 5 ft. apart, along the column. The wind loads normal to the roof were concentrated at the panel points of the truss. The intensity of the roof loads was taken in accordance with Duchemin's formula. The reader is cautioned against drawing too general conclusions from the data given in Table III; for only a limited number of structures was analyzed, and all trusses were of the same type and slope. The analysis was made for columns pin-connected at their bases, and rigidly fixed at their bases. The results given in Table III are ratios, and are thus explained.

H = wind load on windward or left column, exclusive of the half-panel load at the base, plus horizontal component of the normal roof load.

H_R = horizontal reaction at base of leeward or right column.

d = distance from base of column to foot of knee-brace.

x = distance from base of column to point of contraflexure of windward column.

y = distance from base of column to point of contraflexure of leeward column.

Suppose that $H = 10{,}000$ lb. for the structure, in which $l = 40$ ft. and $h = 21$ ft. If the columns are pin-connected, the horizontal reaction at the base of the leeward column is

$$H_R = 10{,}000 \times 0.385 = 3{,}850 \text{ lb.}$$

If the columns are fixed and $d = 15$, then

$$H_R = 10{,}000 \times 0.356 = 3{,}560 \text{ lb.}$$
$$x = 15 \times 0.488 = 7.32 \text{ ft.}$$
$$y = 15 \times 0.581 = 8.71 \text{ ft.}$$

The structure is statically determinate when one horizontal reaction and the points of contraflexure are known.

The columns are seldom, if ever, built with pin connections; nor are they sufficiently anchored to the masonry to insure a perfectly rigid connection. The actual conditions existing

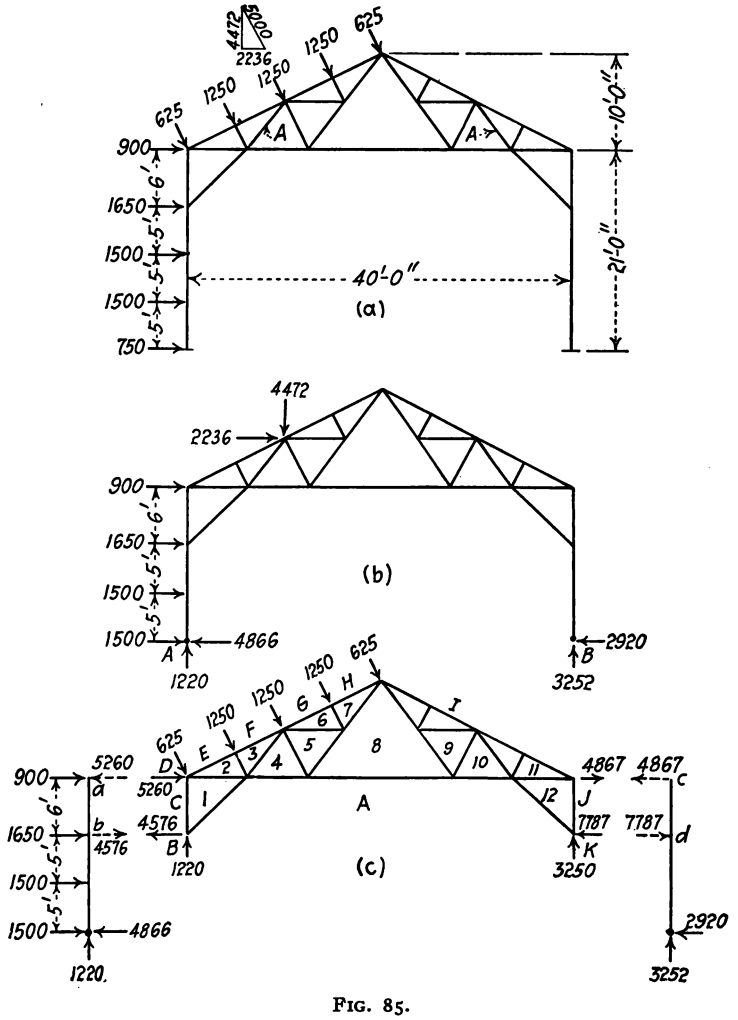


FIG. 85.

in all mill-building bents are to be found between these two limits. For example, in the structure before mentioned, where $l = 40$ ft. and $h = 21$ ft., the horizontal reaction of the leeward column will be between 0.385 and 0.356 of the hori-

zontal load. The point of contraflexure, or point of zero moment, in the leeward column will be found between the base of the column and the point 0.488 of the distance to the foot of the knee-brace. A comparison of the data in columns 4 and 5 of Table III indicates that $0.375H$, instead of $0.5H$, is the more reasonable estimate of the horizontal reaction of the leeward column; when the columns are considered partially fixed. The data given in columns 6 and 7 indicates that the point of contraflexure in each column may be taken at one-third

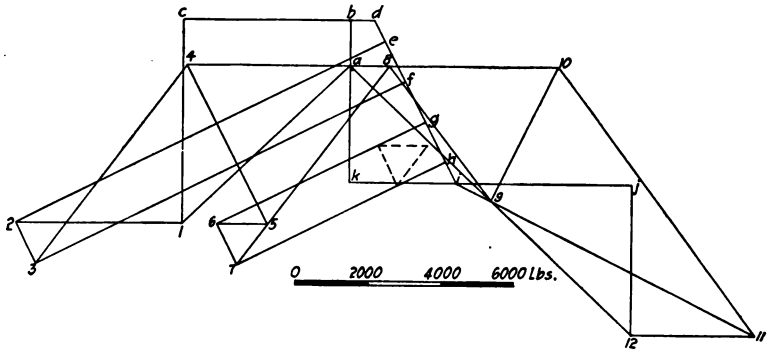


FIG. 86.

the distance from the base of the column to the foot of the knee-brace. This conclusion is justified only when the masonry is sufficiently massive to resist the moment thereby alloted to it.

The wind concentrations shown in Fig. 85*a* were determined from the data given in Article **76.** The sum of the horizontal loads, exclusive of the half-panel load of 750 lb., is

$$H = 7{,}786$$

If the horizontal reaction of the leeward column is taken as three-eighths of the total load H; then

$$H_R = 0.375 \times 7{,}786 = 2{,}920$$
$$H_L = 7{,}786 - 2{,}920 = 4{,}866$$

The point of contraflexure in each column will be taken one-third the distance from the column base to the foot of the knee-brace; and the structure is thereby transformed into the

statically determinate frame as shown in Fig. 85*b*, in which the moments are zero at *A* and *B*.

Moments about *A*		Moments about *B*	
1,500 × 5 =	7,500	4,472 × 30 =	134,160
1,650 × 10 =	16,500	2,236 × 21 =	46,956
900 × 16 =	14,400	900 × 16 =	14,400
2,236 × 21 =	46,956	1,650 × 10 =	16,500
4,472 × 10 =	44,720	1,500 × 5 =	7,500 85,356
40)130,086		40)48,804	
3,252 = V_B		1,220 = V	

The truss and columns are shown separately in Fig. 85*c*. The known forces acting on the columns are shown by full lines. The horizontal forces, represented by dotted lines at *a*, *b*, *c* and *d*, which the truss must exert on the columns in order to hold them in equilibrium, are determined as follows:

Moments about *c*

$$\text{force at } d = \frac{2{,}920 \times 16}{6} = 7{,}787 \text{ acting to the right.}$$

$$\text{force at } c = 7{,}787 - 2{,}920 = 4{,}867 \text{ acting to the left.}$$

Moments about *a*

4,866 × 16 =		77,856
1,650 × 6 =	9,900	
1,500 × 11 =	16,500	
1,500 × 16 =	24,000	50,400
	6)27,456	
	4,576 = force at *b* acting to the right.	

force at a = 900 + 1,650 + 4,576 + 1,500 + 1,500 − 4,866 = 5,260 acting to the left.

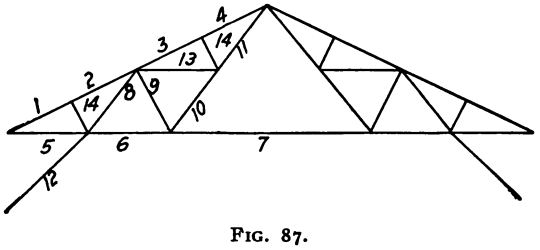


FIG. 87.

TABLE IV.—COMPARISON OF WIND STRESSES—1,000-LB. UNITS

Member	Wall bearing truss				Mill building truss Case V	Wind load acting vertically, both sides Case VI
	Case I $H_1 = 2{,}236$ $H_2 = 0$	Case II $H_1 = 0$ $H_2 = 2{,}236$	Case III $H_1 = 1{,}118$ $H_2 = 1{,}118$	Case IV $H_1 = 1{,}537$ $H_2 = 699$		
1	−5.6	−5.6	−5.6	−5.6	−11.3 + 9.3	−9.8
2	−5.6	−5.6	−5.6	−5.6	−11.3 + 9.3	−9.3
3	−5.6	−5.6	−5.6	−5.6	− 6.4 + 1.1	−8.7
4	−5.6	−5.6	−5.6	−5.6	− 6.4 + 1.1	−8.2
5	+7.0 +2.8	+4.8	+5.9	+6.3	+ 4.6 − 3.5	+8.8
6	+5.6 +2.8	+3.4	+4.5	+4.9	+ 4.5 − 5.7	+7.6
7	+2.8	+0.6	+1.7	+2.1	− 1.1	+5.0
8	+1.4	+1.4	+1.4	+1.4	+ 6.9 − 9.3	+1.3
9	−2.5	−2.5	−2.5	−2.5	− 5.0 + 4.2	−2.3
10	+2.8	+2.8	+2.8	+2.8	+ 5.5 − 4.7	+2.5
11	+4.2	+4.2	+4.2	+4.2	+ 6.9 − 4.7	+3.8
12	0	0	0	0	+ 6.3 −10.8	0
13	+1.4	+1.4	+1.4	+1.4	+ 1.4	+1.3
14	−1.3	−1.3	−1.3	−1.3	− 1.3	−1.1

Since the forces which the truss exerts on the columns must be respectively equal in magnitude, but opposite in sense, to the forces which the columns exert on the truss; the forces at *a*, *b*, and *d* are therefore shown reversed on the truss. These forces virtually represent the bending effect imposed by the columns on the truss. The vertical reactions, or the thrust of the columns on the truss, are also transferred to the truss. The truss is now in equilibrium under the action of the forces shown thereon, and a stress diagram may be drawn as shown in Fig. 86. The stresses are recorded as Case V in Table IV. The members are numbered to correspond with Fig. 87.

78. Stress Diagram for Wind Unnecessary.—The stresses for Cases I to V inclusive (as recorded in Table IV) were determined for a wind load of 20 lb. per square foot, on a vertical surface; and 14.9 lb. per square foot., or 1,250 lb. per panel acting normal to the roof and on only one side of it. If a wind load of 1,250 lb. per panel is assumed to act *vertically* on *both* sides of the truss in Fig. 85*a;* the resulting stresses will be as given under Case VI in Table IV. It is clear that the stresses in all chord members are greater for Case VI than for Cases I to IV inclusive; and that the slight excess of the stresses in the web members for Cases I to IV are negligible when the additional stresses for dead and snow loads are taken into consideration. In comparing Case V with Case VI, we find that the member 8 requires the most consideration. Members 1 and 2 are the only chord members which have a stress greater in Case V than in Case VI.

It is clear that for wall bearing trusses, a load of $v = 14.9$ lb. per square foot of roof surface taken vertically on both sides of a Fink truss, having a roof slope of 6 in. to 1 ft., will suffice for a wind horizontal load of $w = 20$ lb. per square foot on a vertical surface. It is also clear that v varies directly as w. Likewise, if w remains constant and the slope varies, the corresponding value of v may be taken from Table II of Article **75**. In a wall bearing Warren truss, having a roof slope not greater than 1½ in. to 1 ft., a vertical load of 5 lb. per square foot seems to be an ample allowance for wind.

When the truss is knee-braced to columns, the wind load

may without serious error, be considered vertical and included with the dead and snow loads, if special consideration is given to the columns, knee-braces and the web members which are connected to the knee-braces and nearest in line with them. It is probably true that a stress diagram for wind is drawn for not more than one truss for every hundred that are designed.

79. Combined Snow and Wind Loads.—A considerable amount of speculation has been made regarding the proper combination of stresses caused by wind and snow. It seems reasonable to predict that in a given locality the maximum wind and snow loads will not occur simultaneously. If the snow is not covered by a crust, a high wind will blow it from the roof. If the roof is nearly flat, the snow load may be large; but the wind stresses will be small, especially in a wall bearing truss. On the other hand, the wind stresses are relatively larger on a steep roof, but little snow can be retained.

For buildings in the latitudes of New York and Chicago, it is the general practice to consider a load of 25 lb. per square foot of roof surface acting vertically, as sufficient for combined wind and snow.

80. Design Stresses.—The following data will be assumed: Wall bearing trusses (Fig. 83*a*) 15 ft. apart; span, 40 ft., rise 10 ft., roof covering including purlins weighs 10 lb. per square foot; combined snow and wind assumed at 25 lb. per square foot of roof surface. The length of the top chord is 22.36 ft. The total weight on the truss is $35 \times 15 \times 2 \times 22.36 = 23{,}500$ lb., or 590 lb. per foot of span. From Table I the assumed weight of the truss is 1,500 lb. The total load is 25,000 lb. or 3,125 lb. per panel. A stress diagram drawn for a vertical load of 3,125 b., placed at each of the seven joints of the top chord will determine the design stresses. The half-panel load at each support has no influence upon the stresses.

When the same truss is supported on columns, as in Fig. 85*a*, the stress diagram is drawn in the same manner, using the same panel loads as for the wall-bearing truss. All members are designed as for a wall-bearing truss, except the member *A*. The size of one angle necessary to carry the stress, as given by

the stress diagram, is determined; and two angles of this size are used.

The horizontal wind load on the side of the building, and the horizontal component of the normal wind load on the roof, which are carried by each bent are determined; and three-eights of the total is taken as the horizontal reaction of the leeward column. In Article 77 this reaction was found to be 2,920 lb. This reaction, multiplied by two-thirds the height of the column to the foot of the knee-brace, gives the maximum bending moment at the foot of the leeward knee-brace; which is

$$2{,}920 \times 10 = 29{,}200 \text{ ft.-lb.}$$

This bending moment is obviously greater than at any point in the windward column. Each column should be designed to resist the bending moment of 29,200 ft.-lb. and a direct compression of 12,500 lb.

In designing the knee-braces, the leeward knee-brace should be considered. The horizontal component of the stress, easily determined from the sketch of the leeward column in Fig. 85*c*, is 7,787 lb.; from which the compressive stress may be quickly found.

81. Conclusions.—Several years ago a purchaser wrote to a structural steel company for the price of a "steel building, 40 ft. wide, 100 ft. long and 2 miles from the railroad station." The author, who was at the time a designer for the company, was given the letter and instructed to make a "design and estimate." While the typical specification written by a purchasing agent gives more detail than the one cited; it is nevertheless true that the cases are rare where competitive designs submitted in accordance with the same specifications are actually made on the same basis. In some designs no bending moment whatever would be provided for in the columns, in others bending moment would be considered on the basis of 16,000 lb. per square inch reduced for compression; while in others 24,000 lb. per square inch might be used. But the most interesting feature would be found in a comparison of the designs of knee-braces and bracing which are generally a matter of opinion with each designer; and his opinion may vary with the hour of the day or the day of the week.

There is no generally accepted standard specification for buildings in America. Until one has been written, the treatment of stresses as given in Article **80**, though not strictly scientific, should not be condemned by any theorist who knows nothing of the perplexities of the designer in playing the role of a mind reader while interpreting the average building specifications.

CHAPTER IV

BRIDGES

SEC. I. STANDARD TYPES

82. Standard Types.—The most common types of railway steel bridges are shown in Figs. 88 to 94. In the deck plate-girder type the cross-ties which support the rails rest directly on the top flanges of two girders, placed from 6 to 7 ft. apart

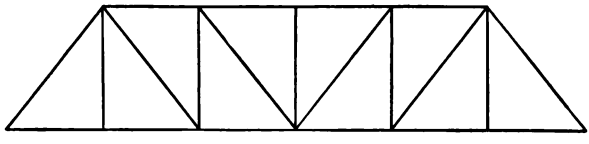


FIG. 88.—Pratt Truss.

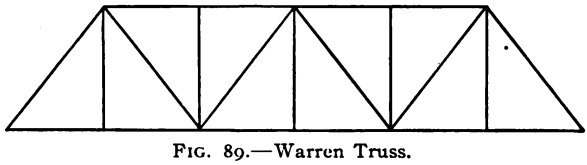


FIG. 89.—Warren Truss.

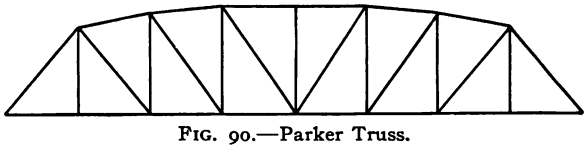


FIG. 90.—Parker Truss.

In the through plate-girder type the cross-ties rest on short deck beams or girders which are supported at each end by cross girders, usually called floor beams, which are in turn supported by the main girders. The through type is more expensive than the deck type, and is never used unless the distance from the rail to the under side of the main girders is so small that deck span is impossible. Plate-girders are used for spans up to about 115 ft.

The Pratt truss (Fig. 88) is the common type for pin-connected trusses in which the diagonal web members are eye-bars, or other slender members designed to take tension only.

The Warren truss (Fig. 89) is built with stiff web members, capable of resisting alternate stresses of tension and compression. The Pratt and Warren types are used for spans between about 115 ft. and 200 ft.

The Parker truss (Fig. 90), more often called a curved chord truss or a came'-back truss, and often having eye-bars in the bottom chord and diagonal web members, is the standard type for spans between about 200 ft. and 300 ft.

The Baltimore truss is shown in Figs. 91 and 92. The eco-

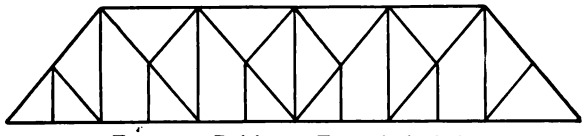


FIG. 91.—Baltimore Truss (sub-ties).

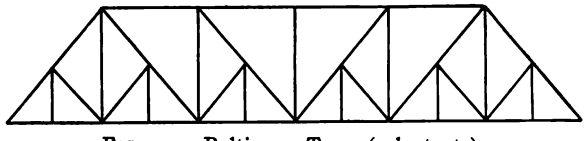


FIG. 92.—Baltimore Truss (sub-struts).

nomic height of a bridge, being from one-fifth to one-eighth of the span, increases with the length. The economic panel length does not increase in the same ratio, consequently as the height increases the diagonals become steeper. From a practical as well as a theoretical standpoint, the diagonals should not exceed a slope of about 3 vertical in 2 horizontal. In order to meet this requirement in large spans, the truss may be built with subdivided panels as in the Baltimore truss. The floor beams in the middle of a long panel are supported by a *sub-vertical*, connecting to the main diagonal and a sub-tie as in Fig. 92 or a sub-strut as in Fig. 93. Theoretically the Baltimore truss has no standing, for if the length of span makes sub-paneling essential, a parallel chord truss is not economical.

The Pennsylvania truss with sub-ties (Fig. 93) and substruts (Fig. 94) is used for all simple truss spans over apprximately 300 ft. in length. The Metropolis bridge over the Ohio River, 720 ft. long, is the longest span of this type which has been built to date.

The standard types here mentioned are also employed in highway bridges. Trusses are frequently used in highway bridges for spans as short as 40 or 50 ft.

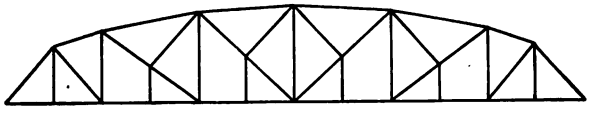


FIG. 93.—Pennsylvania Truss (sub-ties).

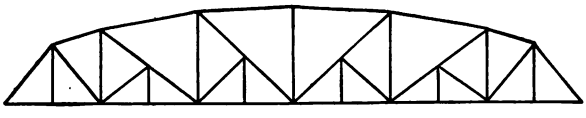


FIG. 94.—Pennsylvania truss (sub-struts).

83. Moving Loads.—The loads which a roof truss supports are usually considered stationary; and stress analysis proceeds directly after the determination of the load and reactions. The process is not so simple in the case of bridges. The loads *move* upon the structure, and the shear or bending moment at any section, or the stress in any member, varying with the movement, is required only for that particular position of the loads which will cause the maximum shear, bending moment or stress. The first task, therefore, is the development of a method for finding the required position of the moving loads in any given case. This leads us to a consideration of "Influence Lines."

SEC. II. DECK BEAMS AND GIRDERS

84. Influence Line for Reaction.—Let P (Fig. 95) represent a load of 1 lb., moving upon the beam AB. Draw the line QO, intersecting the verticals through A and B. Lay off $QK = 1$ lb., draw the line OK, and let IJ represent the ordinate under the load P. As P moves from B to A, the reaction at A in-

creases uniformly from zero, when the load is at B; to 1 lb., when the load reaches A. Likewise the ordinate IJ increases uniformly from zero at O to QK = 1 lb., at Q. Hence for any position of the load, the ordinate IJ, directly under the load, represents the reaction at A. The line OK is called an influence line for the reaction at A, since the ordinate IJ shows how the reaction at A varies or is influenced by any movement of the load.

The influence line, thus drawn by the aid of the unit load, may be used for finding the reaction at A for a series of loads in any position by multiplying the ordinate under each load by the ratio of its weight to 1 lb., and adding the products. In Fig. 96 the ordinates for the loads Q and S are 0.7 and 0.4

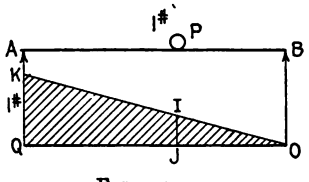


FIG. 95.

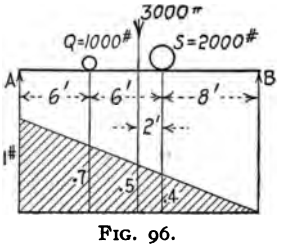


FIG. 96.

respectively. Hence, the reaction at A is

$$2{,}000 \times 0.4 = 800$$
$$1{,}000 \times 0.7 = 700$$
$$1{,}500 \text{ lb.}$$

Since the influence line has a constant slope, the reaction at A may also be determined by finding the product of the total load (3,000), and the ordinate at the center of gravity of the loads; thus,

$$3{,}000 \times 0.5 = 1{,}500 \text{ lb.}$$

The load of 1 lb., which is invariably used in constructing influence lines, may be considered as acting the role of a scout, sent across the span in advance of other loads. The effect of this advance load upon the reaction, when represented graphically, becomes an influence line for the reaction. The same is true in the case of shear or bending movement. Influ-

ence lines thus drawn may be used as a basis for determining the effect produced by a series of loads; such as a road roller, traction engine, or a railway train.

85. Influence Line for Shear in a Beam.—Let us first have a clear understanding of what we mean by shear.

The shear at any normal section of a beam is the algebraic sum of all the *transverse* forces acting on one side (either side) of the section. When this sum or resultant force acts upward on the left of the section, the shear is positive. Obviously, if the resultant of the forces on the left of the section acts upward, the resultant of the forces on the right of the section acts downward.

The influence line for shear at section C (Fig. 97) is desired.

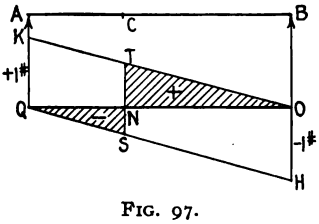


FIG. 97.

When the load of 1 lb. is between B and C, the only force acting on the left of the section is the reaction at A. Hence, the influence line OK for the left reaction, is also the influence line for shear at C, when the load is between B and C.

When the load of 1 lb. is between A and C, the only force on the right of the section is the right reaction acting upward; and the shear at the section is the right reaction taken negatively. Hence, the line QH, being the influence line for the right reaction taken negatively, is the influence line for shear at C when the load of 1 lb. is between A and C. Therefore the line $QSNTO$ is the influence line for shear on the section at C; and shows that when a single load (whether 1 lb. or 20 tons) crosses the span from B to A, the shear at C increases uniformly until the load is just at the right of C. When the load crosses the point C, the shear at C instantly becomes negative, increasing to zero when the load arrives at A.

86. Influence Line for Bending Moment in a Beam.—The bending moment at any normal section of a beam is the algebraic sum of the moments of all the forces acting on one side (either side) of the section, taken about the center of gravity of the section as an axis. When this sum or resulting moment

is clockwise on the left of and about the section, the bending moment is positive. Obviously if the sum of the moments on the left of the section is clockwise, the sum of the moments on the right of the section is counter-clockwise.

The influence line for the bending moment at section C (Fig. 98) is desired. When the load of 1 lb. is between B and C, the only force acting on the left of the section is the left reaction at A; and the bending moment at C equals 10 ft. times the left reaction. As the load moves from B to A, the left reaction increases uniformly from 0 to 1 lb., and 10 ft. times the left reaction increases uniformly from 0 to 10 ft.-lb. Likewise the ordinate to the line OK, increasing uniformly from zero at O to 10 ft.-lb. at Q, represents 10 ft. times the left reaction. Hence the line ON is the influence line for the bending moment at C when the load of 1 lb. is between B and C. The ordinates are positive, since the moment of the force on the left of the section is clockwise.

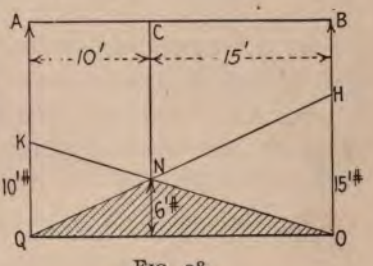


FIG. 98.

When the load of 1 lb. is between C and A, the only force acting on the right of the section is the right reaction at B; and the moment at C equals 15 ft. times the right reaction. As the load moves from A to C, the right reaction increases uniformly from 0 to 1 lb.; and 15 ft. times the right reaction increases uniformly from 0 to 15 ft.-lb. Likewise the ordinate of the line QH, increasing uniformly from zero at Q to 15 ft.-lb. at O, represents 15 ft. times the right reaction. Hence, the line QN is the influence line for the bending moment at C, when the load of 1 lb. is between A and C. These ordinates are also positive, since the moment of the force on the right of the section is counter-clockwise. Therefore, the line QNO is the influence line for the bending moment at C; and shows that when a single load (whether 1 lb. or 100 lb.) crosses the span from B to A, the bending moment at C increases uniformly to a maximum as the load arrives at C, and decreases uniformly to zero as the load arrives at A.

87. Criterion for Maximum Bending Moment.—Consider a series of loads so connected that the distances between loads are maintained (Fig. 99). As these loads move on the span, the bending moment at C varies. It is desired to find the position of these loads on the span, so that the bending moment at C will be a maximum. The influence line QNO is drawn in accordance with the preceding article. The bending moment at C, for any given position of the loads, may be determined by multiplying each load by its corresponding ordinate in the influence line diagram; and finding the sum of these products. Or, since the influence line has a constant slope from Q to N, and a constant slope from N to O; the bending moment at C may be expressed by the equation

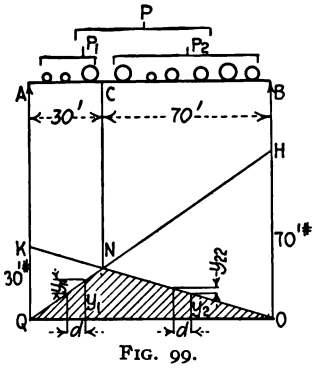


FIG. 99.

$$M_C = P_2y_2 + P_1y_1$$

in which P_1 = the sum of the loads between A and C
P_2 = the sum of the loads between C and B
y_1 = the ordinate at the resultant P_1
and y_2 = the ordinate at the resultant P_2.

Let the loads be moved any small distance d to the left; in such a way that no loads pass the points A, C and B. In other words P_1 and P_2 remain constant while the movement takes place. The bending moment at C for this new position is

$$M'_C = P_2(y_2 + y_{22}) + P_1(y_1 - y_{11}).$$

The change in the bending moment at C is

$$\Delta M_C = M'_C - M_C = P_2y_{22} - P_1y_{11}$$

Has the bending moment at C been increased or diminished by this movement? The answer to this question depends upon whether

$$P_1y_{11} \lessgtr P_2y_{22}.$$

As long as $P_1y_{11} < P_2y_{22}$

the bending moment at C is increased by the movement of the loads to the left.

From similar triangles

and $$y_{11}:d::70:100$$
$$y_{22}:d::30:100$$

or $$y_{11}=\frac{70d}{100} \text{ and } y_{22}=\frac{30d}{100}.$$

Hence the bending moment at C will increase as long as

$$\frac{70d}{100}P_1<\frac{30d}{100}P_2$$

or as long as $$7P_1<3P_2.$$

Let $P_1+P_2=P=$ total load on the span.

If $$7P_1<3P_2$$

and $$3P_1=3P_1$$

then $$10P_1<3P$$

or $$P_1<\frac{3}{10}P.$$

Hence, the bending moment at C will increase by a movement of the loads to the left as long as

$$P_1<\frac{3}{10}P$$

If no additional loads move on to the span at B as the loads move to the left, and none pass off at A, the total load P on the span remains constant; but the loads P_1 on the segment AC are increased whenever a load passes the point C.

Now there will be a certain load which, when approaching C, will render

$$P_1<\frac{3}{10}P,$$

and the bending moment at C will be increasing; but, having passed C and become a part of P_1, will render

$$P_1>\frac{3}{10}P,$$

in which case the bending moment at C will be decreasing. This particular load is called the *critical load*, for when it is at C, the bending moment at C is a maximum.

MOMENT TABLE

2-142 Ton Engines followed by 4000 lb. per ft.

COOPER'S CLASS E-40

a	284	274	254	234	214	194	174	154	135	116	98	82	68	56	46	38	32	28	26	26	28	32	38	46	56	68	82	98	116
b	10	30	50	70	90	108	126	142	158	172	186	198	208	216	222	226	228	228	226	222	216	208	198	186	172	158	142	126	108
c	10	20	20	20	20	18	13	13	13	10	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	18	13
d	8′	5′	5′	5′	9′	5′	6′	5′	8′	5′	5′	5′	5′	9′	5′	6′	6′	5′	5′	5′	6′	6′	5′	9′	5′	5′	5′	5′	5′
e	0	8	13	13	23	32	37	43	48	56	64	69	74	79	83	88	93	93	90	104	109								
f	109	101	96	91	86	77	72	66	61	53	45	40	35	30	25	21	16	10	5	0									
	16361	15274	13954	11394	9514	7794	6793	5557	4569	4206	3670	2776	1976	1276	676	403	195	65											
(13)	14944	13904	11984	10164	8444	6824	5883	5017	4224	3496	3016	2216	1516	916	416	203	65												
(12)	13569	12599	10779	9069	7439	5919	5043	4242	3514	2851	2421	1721	1121	621	221	78													
(11)	12041	11111	9411	7811	6311	4911	4118	3390	2740	2165	1785	1205	725	345	65														
(10)	10346	9606	8386	6836	5436	4136	3408	2745	2160	1640	1320	840	460	180															
(9)	8728	7938	6618	5168	3978	2858	2247	1701	1283	880	600	300	100																
(8)	7683	6928	5608	4388	3268	2248	1702	1221	818	480	300	100																	
(7)	6708	6018	4708	3678	2658	1788	1257	841	503	230	100																		
(6)	5948	5308	4088	3068	2148	1328	912	561	288	80																			
(5)	4682	4072	3112	2252	1472	832	520	273	104																				
(4)	3496	3016	2216	1516	916	416	208	65																					
(3)	2851	2421	1721	1121	621	221	78																						
(2)	2155	1765	1205	725	345	65		78	221	411	961	1691	2331	3171	3831	4092	5368	6239											
(1)	1640	1320	840	460	180		65	208	416	656	1296	2036	2676	3816	4544	5357	6208	7144											
(0)	836	600	300	100		117	299	559	884	1214	2034	2954	3974	5094	5939	6949	7837	8890											
(1)	480	300	100		100	282	520	854	1244	1624	2544	3564	4684	5904	6814	7789	8842	9960											
(2)	230	100		100	300	547	859	1249	1704	2134	3154	4274	5494	6814	7769	8829	9947	11130											
(3)	80		100	300	600	912	1259	1744	2264	2744	3864	5084	6404	7824	8864	9969	11162	12400											
(4)		100	300	600	1240	1656	2137	2696	3329	3830	5160	6540	8020	9600	10744	11963	13240	14592											
(5)	160	420	780	1240																									

9000 lb. E.L.F.

Moments in 1000 ft. lb. for One Truss

FIG. 100.

Thus we see how an influence-line diagram may be used to develop the criterion for maximum bending moment at any section of a beam of any length. After the correct position of the loads has been determined, the maximum bending moment may be computed by scaling the length of the ordinate under each load; and taking the sum of the products of each load and its corresponding ordinate.

88. Cooper's Standard Train Loads.—When the series of loads represents the wheel concentrations of a railway train, the bending moment may be more easily computed by the help of a "Momemt Table"; as shown in Fig. 100. This table is a collection of data for what is known as a "Cooper's E-40 Loading"; consisting of two 142-ton locomotives, followed by a uniform train load of 4,000 lb. per linear foot. Each axle for the large wheel supports a load of 40,000 lb. A train is usually supported by two girders or trusses, one-half of the load coming upon each. For this reason, wheel loads or the loads supported by each half of the structure, instead of axle loads, are given in the table. The wheel loads are given on line *c*; wheel 1 supports 10,000 lb.; wheel 4, 20,000 lb.; wheel 16, 13,000 lb.; etc. Accumulative loads are given on lines *a* and *b*. The spacing of the wheels is given on line *d*, and the accumulative distances on lines *e* and *f*. The quantities given below line *f* are moments expressed in units of 1,000 ft.-lb.

Suppose the moment is required of all the wheels to the left of, and about wheel 6. Follow the space between wheels 5 and 6 down to the heavy line; and then to the left, under wheel 1, read 1,640. The moment of the wheels up to, and including wheel 3, about wheel 6 is 840. Verify these quantities by computation. The quantities below the heavy zigzag line are moments on the right of various wheels, instead of on the left.

Cooper's Loadings are specified more extensively in American railway practice than in any other. They are divided into classes and are specified as Cooper's E-40, E-50, E-55, etc. The wheel spacing in all classes is the same. A constant ratio exists between the loads of one class and the corresponding loads of any other class. Thus the loads of a Cooper's E-60

are 50 per cent greater than those for an E-40, given in Fig. 100. If an E-60 is specified, the table of Fig. 100 may still be used, if the final results are increased by 50 per cent.

89. Illustrative Problem.

Determine the maximum bending moment for an E-40 loading at the center of a 62-ft. span. Draw the influence line diagram and develop the criterion for maximum bending at point C (Fig. 101). When developed, this criterion will show that, as the train comes on the span at B, the bending moment at C will increase as long as the loads on AC are less than one-half the total load on the span. The influence line shows that as many loads as possible should be placed on the span, with the heavy loads usually (not always) near C, where the ordinates are long.

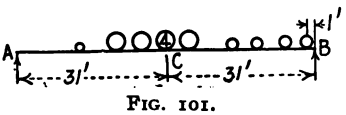


FIG. 101.

Try wheel 4 at C. Wheel 4 is 18 ft. from the head of the train, hence the length of train on the span is $18 + 31 = 49$ ft. Wheel 9, being 48 ft. from the head of the train, is 1 ft. on the span.

$2)142$ = total load on the span
71 = one-half the load on the span
50 = load on AC when wheel 4 is approaching C
70 = load on AC when wheel 4 has passed C.

$$71 \begin{matrix} > 50 \\ > 70 \end{matrix}$$

In either case the load on AC is less than one-half the load on the span, and the bending moment at C continues to increase as wheel 4 crosses C.

Try wheel 5 at C.

Make a sketch similar to Fig. 101 for this, and each succeeding case. In this arrangement the large wheels are not so well located about C; but the loads as a whole are nearer the center of the span.

$2)142$
$71 > 70$ when wheel 5 is approaching C
< 90 when wheel 5 has passed C.

When wheel 5 is approaching C, the load on AC is less than one-

half the load on the span, and the bending moment at C is consequently increasing; but when wheel 5 has passed C, the load on AC is greater than one-half the load on the span, and the bending moment at C is consequently decreasing. Hence wheel 5 at C satisfies the criterion for maximum bending moment at C.

Try wheel 6 at C.

Wheel 1 has now moved off the span, and wheel 10 has come on; causing no change in the total load on the span.

$$2)142$$

$71 < 80$ bending moment at C is decreasing, with wheel 6 approaching C.

< 93 bending moment at C is decreasing, with wheel 6 passing C.

It is to be noted that, as the loads move to the left on the span, the bending moment at C increases from zero, when wheel 1 is at B; to a maximum, when wheel 5 is at C. As wheel 5 passes C, the bending moment at C begins to decrease. To show this, the bending moment at C will now be computed for the three positions of the load when wheels 4, 5 and 6 are at C.

Wheel 4 at C.

We shall compute the bending moment at C by taking the algebraic sum of the moments of all the forces acting on the left of C. These forces are the reaction at A and wheels 1, 2 and 3. The work is greatly simplified by using the moment table. We shall first find the moment of all the wheels about B. The moment of all the wheels to the left of, and about 9, is 3,496 as given in the table. The moment of all the wheels about B equals the moment of all the wheels about wheel 9; plus the weight of all the wheels times 1 ft.

$$142 \times 1 = \begin{array}{r} 3{,}496 \\ 142 \\ \hline 3{,}638 \end{array} = \text{the moment of all the loads on the span about } B.$$

The reaction at A is

$$R_A = \frac{3{,}638}{62}$$

The moment of all the wheels to the left of, and about C is 480, hence the bending moment at C is

$$\frac{3{,}638}{62} \times 31 = 1{,}819$$
$$480$$
$$1{,}339 = \text{bending moment at } C.$$

Wheel 5 at C.

$$3{,}496$$
$$142 \times 6 = 852$$
$$4{,}348 = \text{moment about } B.$$
$$\frac{4{,}348}{62} \times 31 = 2{,}174$$
$$830$$
$$1{,}344 = \text{bending moment at } C.$$

Wheel 6 at C.

$$4{,}072 = \text{moment about wheel 10 of wheels up to and including wheel 2}$$
$$142 \times 7 = 994$$
$$2)5{,}066$$
$$2{,}533 = \text{moment of the left reaction about } C.$$
$$1{,}320 = \text{moment about wheel 6 of wheels up to and including wheel 2.}$$
$$1{,}213 = \text{bending moment at } C.$$

The results are tabulated below:

At C	Bending Moment
Wheel 4	1,339 increasing
Wheel 5	1,344 *a* maximum
Wheel 6	1,213 decreasing

Let us continue the investigation further.

Try wheel 7 at C.

$$2)162$$
$$81 < 93 \text{ decreasing}$$
$$< 106 \text{ decreasing}$$

Try wheel 8 at C.

Note that when wheel 8 is approaching C, wheel 13 is not on the

span and the total load is 162; but when wheel 8 has passed C, wheel 13 is on the span, and the total load is 182; hence,

$$\begin{array}{l} 2)162 \\ \quad 81 < 86 \text{ decreasing} \\ 2)182 \\ \quad 91 < 99 \text{ decreasing.} \end{array}$$

Try wheel 9 at C.

$$\left.\begin{array}{l} 2)162 \\ \quad 81 > 79 \text{ increasing} \\ 2)182 \\ \quad 91 < 92 \text{ decreasing} \end{array}\right\} a \text{ maximum}$$

Since the bending moment was decreasing when wheel 8 had passed C, and was increasing when wheel 9 was arriving at C; there must have been some intermediate position, at which the bending moment was a minimum. This position occurs when wheel 3 leaves the span at A.

Wheel 3 at A.

$$\left.\begin{array}{l} 2)182 \\ \quad 91 < 99, \text{ wheel 3 approaching } A, \text{ decreasing} \\ 2)162 \\ \quad 81 > 79, \text{ wheel 3 passing } A, \text{ increasing} \end{array}\right\} a \text{ minimum}$$

Wheel 10 at C.

$$\begin{array}{l} 2)142 \\ \left.\begin{array}{l} 71 > 52 \\ \quad > 62 \end{array}\right\} \text{increasing} \end{array}$$

Wheel 4 at A.

$$\left.\begin{array}{l} 2)182 \\ \quad 91 < 92 \text{ decreasing} \\ 2)162 \\ \quad 81 > 72 \text{ increasing} \end{array}\right\} a \text{ minimum.}$$

Wheel 11 at C.

$$\begin{array}{l} 2)155 \\ \left.\begin{array}{l} 77 \cdot 5 > 49 \\ \quad > 69 \end{array}\right\} \text{increasing} \end{array}$$

Wheel 12 at C.

$$2)155$$

$$77.5 > 56 \\ > 76 \Big\} \text{ increasing}$$

Wheel 13 at C.

$$2)168$$

$$84 > 76 \text{ increasing}$$

$$2)155$$

$$77.5 < 83 \text{ decreasing} \Big\} a \text{ maximum.}$$

Wheel 14 at C.

$$2)157$$

$$78.5 < 83 \text{ decreasing}$$

$$2)144$$

$$72 < 90 \text{ decreasing.}$$

Check the bending moments at C for the several positions of the train, given in the following table:

At C	At A	Bending Moment in Units of 1000 Ft.-lb.
Wheel 4		1,339
5		1,344 *a* maximum
6		1,213
7		1,143
8		1,083
	Wheel 3	1,075 *a* minimum
9		1,083 *a* maximum
	Wheel 4	1,082 *a* minimum
10		1,165
11		1,302
12		1,357.5
13		1,371.5 *the* maximum
14		1,344.5

Wheels 5, 9 and 13 at C satisfy the criterion for a maximum bending moment at C; and wheels 3 and 4 at A satisfy the criterion for a minimum bending moment at C. It frequently happens, as in the present case, that more than one position of the train will satisfy the criterion for maximum bending moment at a given section. For each two consecutive positions, causing a maximum bending moment; there necessarily exists

an intervening position, causing a minimum bending moment. Only the maximum values have a practical significance. When several positions of the train satisfy the criterion for a maximum bending moment, it becomes necessary to compute the bending moment for each position to determine the greatest maximum value.

90. Problems.

Compute the maximum bending moment at the center of a girder having a span of 100 ft., for an E-40 loading.

Draw the influence line, develop the criterion and compute the maximum bending moment at four sections, 10 ft. apart, between the ends and the center. Remember that the train may come on to the span from either end.

91. General Criterion for Maximum Bending Moment.—In Fig. 99 let $AC = a$ and $AB = l$. Draw the influence line for bending moment at C; in which $QK = a$ ft.-lb., and $OH = l - a$ ft.-lb. Develop the criterion, showing that for maximum bending moment at C,

$$P_1 \lessgtr \frac{a}{l} P \tag{1}$$

This criterion is often expressed in the form

$$P_1 = \frac{a}{l} P \tag{2}$$

or

$$\frac{P_1}{a} = \frac{P}{l} = \frac{P_2}{l - a} \tag{3}$$

and may be stated as follows: A maximum bending moment at any point occurs whenever the train is so placed that the load on each segment, divided by the length of the segment, equals the total load on the span, divided by the span length. In other words, the average load per foot on each segment, equals the average load per foot on the span. In interpreting Eqs. (2) and (3), P_1 represents the load on the segment a plus a portion of the load at C; the remainder of the load at C being considered as a part of P_2. If Eqs. (2) and (3) are satisfied when no load is at C, any movement of the train will cause no change in the bending moment at C as long as P_1 and P_2 remain constant.

92. The Point of Greatest Maximum Bending Moment in a Beam.—By comparing the bending moments at the different

points of the 100-ft. girder in the problem of Article 90, the student may conclude that the maximum bending moment at the center of any beam is greater than at any other point. Such a conclusion is usually erroneous. The maximum bending moment at the center of the 62-ft. span, previously considered, was 1,371.5 ft.-lb., when wheel 13 was at the center and wheel 18 was 1 ft. on the span. When wheel 13 is 32.3742 ft. from the right end of the span (Fig. 102), the bending moment under wheel 13 is 1376.2 ft.-lb., which is greater than the bending moment at the center.

Let X (Fig. 103) represent the point of greatest maximum bending moment in a beam. If the influence line for bending

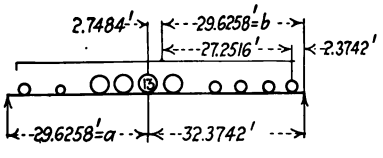


FIG. 102.

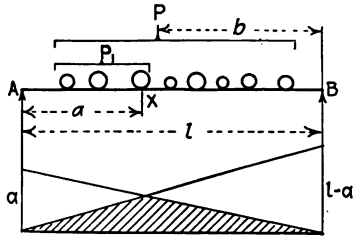


FIG. 103.

moment at X is drawn, and the criterion for maximum bending moment developed, we shall have

$$P_1 \lessgtr \frac{a}{l} P$$

as the wheel at X is considered on the right or the left of X.

Also, if the bending moment is a maximum at X, the shear changes from positive to negative at X. Let R represent the left reaction, then

$$P_1 \lessgtr R$$

as the wheel at X is considered on the right or the left of X. Let b represent the distance from B to the center of gravity of all the loads on the span; then

$$R = \frac{b}{l} P$$

therefore

$$P_1 \lessgtr \frac{b}{l} P$$

whence $$\frac{a}{l} P \lessgtr \frac{b}{l} P$$

or $$a \lessgtr b$$

as the wheel at X is considered on one side or the other of X. Obviously, when the wheel is at X

$$a = b$$

When wheel 13 is at the center of the span (Fig. 102), wheel 8 is over the left support, and consequently not on the span.

$$155)\overline{4{,}224}$$
$$27.2516$$

The center of gravity of wheels 9 to 18 inclusive is 27.2516 ft. from wheel 18; or 2.7484 ft. to the right of wheel 13.

$$62.$$
$$\underline{2.7484}$$
$$2)\overline{59.2516} = a + b$$
$$29.6258 = a = b$$

The center of gravity of the total load is 29.6258 ft. from the right end of the span; and the point of maximum bending moment is under wheel 13, which is 29.6258 ft. from the left end of the span. The maximum bending moment is 1376.2 ft.-lb.

The point of maximum bending moment is at the center of the span only when the center of gravity of the wheels on the span coincides with the wheel which causes the maximum bending moment at the center. Otherwise the point of maximum bending moment is not at the center; but near the center, under the wheel which causes maximum bending moment at the center; when that wheel is as far from one end of the span as the center of gravity of the loads on the span is from the other end.

93. Problems.

A road roller is supported on two axles, 9 ft. apart. The load on the front axle is 10,000 lb. The load on the rear axle is 20,000 lb. Compute the maximum bending moment (*a*) in a span 21 ft. long: (*b*) in a span 13 ft. long. What is the greatest maximum bending moment in a 100-ft. span for an E-40 train? For an E-60 train?

94. Criterion for Maximum Shear in a Beam.—By referring to the influence line for shear (Fig. 97), it is apparent that the positive shear at C increases from zero to a maximum, as wheel 1 comes on the span at B and moves to C. As wheel 1 crosses the point C, the shear at C is instantly diminished by the amount of wheel 1. Evidently a maximum shear occurs as each successive wheel arrives at C, followed by a minimum shear, as the wheel crosses C. When wheel 1 is at the center of a 100-ft. span, the shear at the center equals the left reaction, or 37.8. This shear is instantly decreased to 27.8, as wheel 1 crosses the center. When wheel 2 comes up to the center, the shear is $49.36 - 10 = 39.36$; instantly decreasing to 19.36 as wheel 2 crosses the center; and increasing to 26.96 as wheel 3 arrives at the center. In the ordinary beam or deck plate-girder span, for which the influence for shear in Fig. 97 is typical, the maximum positive shear on any section in the left-half of the span will usually (not always) occur when wheel 2 is at the section.

95. Problem.

Compute the maximum positive shear for an E-40 loading for sections 10 ft. apart, from the center to the left end of a 100-ft. deck plate-girder span.

SEC. III. PARALLEL CHORD TRUSSES

96. A railway truss bridge is shown in Fig. 104, to illustrate the manner in which the wheel loads of a train are carried to and supported by the piers at either end. The wheels rest upon the rails which are supported by cross-ties. The ties rest on longitudinal *stringers* which are supported by cross-beams, more often called *floor beams*. The floor beams are supported by the truss at the bottom chord panel points. *It should be thoroughly understood, and constantly kept in mind, that the truss does not directly support any wheel loads. The truss receives the weight of the train only in the form of floor beam loads, delivered to the truss at the panel points.*

97. Influence Line for Maximum Stress in a Chord Member. Let the truss in Fig. 105 be loaded in any manner, and let M_2 represent the algebraic sum of the moments about L_2

of all the external forces acting on one side (either side) of the section through the panel 1–2. Since the stress in U_1U_2 varies as M_2, it is clear that the stress in U_1U_2 is a maximum, when

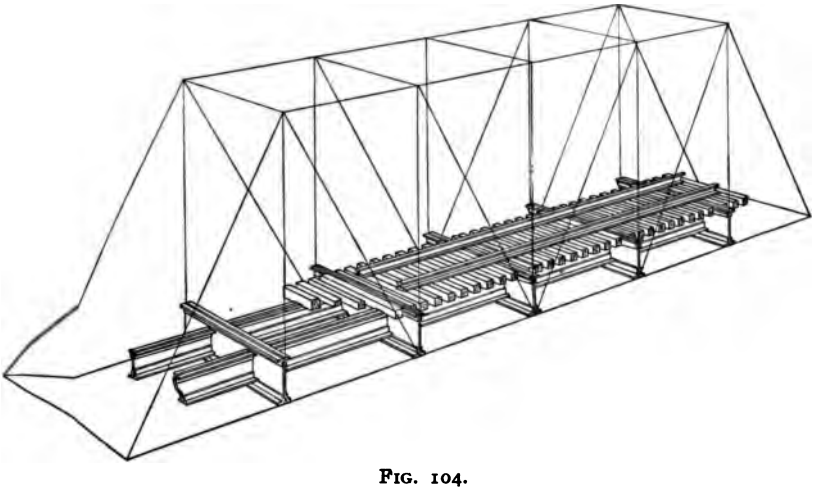


FIG. 104.

M_2 is a maximum. For this reason we shall draw the influence line for M_2 as a load of 1 lb. moves across the span.

When the load of 1 lb. is between L_4 and L_2, the only force acting on the left of the section is the left reaction R_0, and $M_2 = 50R_0$. As the load moves from L_4 to L_0, R_0 increases uniformly from 0 to 1 lb., and $50R_0$ increases uniformly from 0 to 50 ft.-lb. Hence the line ON is the influence line for M_2 when the load of 1 lb. is between L_4 and L_2.

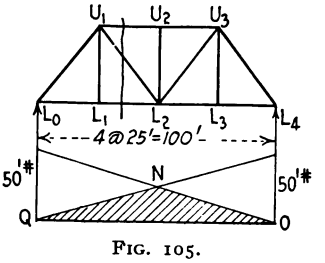


FIG. 105.

When the load of 1 lb. is between L_0 and L_1, the only force acting on the right of the section is the right reaction R_4, and $M_2 = 50R_4$. When the load is between L_1 and L_2, there are two forces acting on the right of the section—a floor beam load at L_2 and the truss reaction R_4; and the moment of these two forces about L_2 is also $M_2 = 50R_4$. As the load moves from L_0 to L_4, R_4 increases uniformly from 0 to 1 lb.; and $50R_4$ increases uniformly from 0 to 50 ft.-lb. Hence, the line QN

is the influence line for M_2, when the load of 1 lb. is between L_0 and L_2.

Now it may be observed that the influence line ONQ for M_2 is identical with the influence line which was drawn, when the criterion for maximum bending moment at the center of the 100-ft. girder was required in Article **90**. Hence, the criterion and position of an E-40 train are the same in each case. It was found in Article **90** that wheels 11, 12 and 13 at the center,

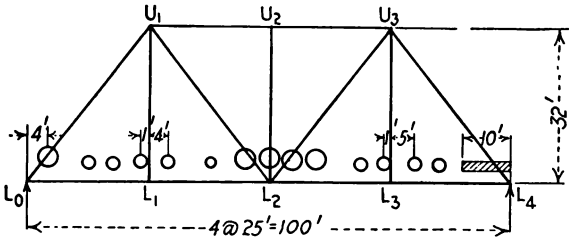


FIG. 106.

satisfied the criterion for a maximum bending moment; wheel 12 giving the greatest maximum, or 3,219 ft.-lb.

98. Illustrative Problem.

Determine the stress in U_1U_2 for an E-40 loading when wheel 12 is at L_2 (Fig. 106).

There are seven forces acting on the truss—five floor beam loads and two truss reactions. After the floor beam loads and the resulting truss reactions have been determined, the stress in U_1U_2 may be obtained by finding the sum of the moments about L_2 of all the forces acting on one side (either side) of a section through panel 1–2, and dividing the sum by 32.

ΣM of the floor beam loads about L_4.

23.80	×	0	= 0
56.56	×	25	= 1,414
74.52	×	50	= 3,726
51.92	×	75	= 3,894
27.20	×	100	= 2,720
234.00		100)	11,754

117.54 = truss reaction at L_0

ΣM about L_2 of the forces on the left of the section.

$$\begin{aligned} 117.54 \times 50 &= 5{,}877 \\ 51.92 \times 25 &= 1{,}298 \\ 27.20 \times 50 &= 1{,}360 \quad 2{,}658 \\ 32)\,3{,}219 & \\ 100.6 &= \text{compressive stress in } U_1U_2. \end{aligned}$$

The problem may be solved more quickly by considering the wheel loads instead of the floor beam loads, and using the moment table.

ΣM about L_4.

$$\begin{aligned} & 9{,}514 \\ 214 \times 10 &= 2{,}140 \\ 10^2 &= 100 \\ 100)\,11{,}754 & \\ 117{\cdot}54 &\text{ truss reaction at } L_0. \end{aligned}$$

I 7.54 × 50 = 5,877 = moment about L_2 of the left reaction
2,658 = moment about wheel 12 of all wheels
32)3,219 from 5 to 12 inclusive.
100.6 = compressive stress in U_1U_2.

99. General Criterion for Maximum Chord Stress.—Let M_2 represent the algebraic sum of the moments about U_2 or L_2 of all the external forces acting on one side (either side) of the section through panel 1–2 of either truss (Fig. 107), as a load of 1 lb. moves across either span. When the load of 1 lb. is between L_6 and L_2, the only force acting on the left of the section is the left reaction R_0, and $M_2 = aR_0$. As the load moves from L_6 to L_0, R_0 increases uniformly from 0 to 1 lb., and $M_2 = aR_0$ increases uniformly from 0 to a ft.-lb. Likewise the ordinate of the line OK, increasing uniformly from 0 at L_6 to a ft.-lb. at L_0, represents aR_0. Hence, the line ON is the influence line for M_2, when the load of 1 lb. is between L_6 and L_2.

When the load of 1 lb. is between L_0 and L_1, the only force acting on the right of the section is the right reaction R_6. When the load is between L_1 and L_2, there are two forces acting on the right of the section—a floor beam load at L_2 and the reaction R_6. Whether the load of 1 lb. is in the first or second

panel, $M_2 = (l - a)R_6$. As the load moves from L_0 to L_6, R_6 increases uniformly from 0 to 1 lb.; and $M_2 = (l - a)R_6$ increases uniformly from 0 to $l - a$ ft.-lb. Likewise the ordinate to the line QH, increasing uniformly from 0 at L_0 to $l - a$

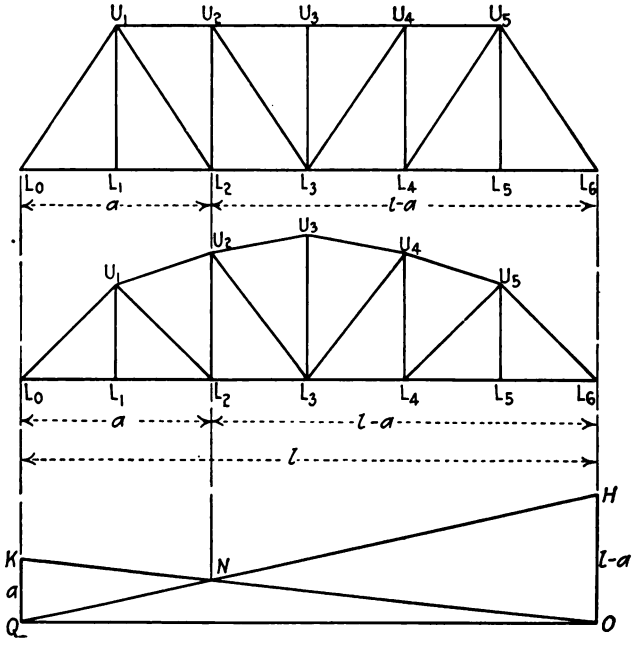


FIG. 107.

ft.-lb. at L_6, represents $(l - a)R_6$. Hence, the line QN is the influence line for M_2, when the load of 1 lb. is between L_0 and L_2.

Let P_1 represent the load on the segment a and let P represent the total load on the span; then the criterion for a maximum M_2, developed as in Article 87, is

$$P_1 \begin{matrix} < \\ > \end{matrix} \frac{a}{l} P$$

The stress in U_1U_2 or in L_2L_3 for either truss, found by dividing M_2 by the proper arm, is a maximum when M_2 is a maximum.

This criterion is often expressed in the form

$$P_1 = \frac{a}{l}P$$

or

$$\frac{P_1}{a} = \frac{P}{l} = \frac{P_2}{l - a}$$

and may be stated as follows: A maximum stress in any chord member of a Pratt or Parker truss; or of any other truss in which the influence line is a triangle (as in Fig. 107), occurs whenever the train is so placed that the average load per foot on each of the two segments into which the center of moments divides the span, equals the average load per foot on the span.

If the trusses in Fig. 107 have six equal panels, one-sixth of the total load on the span is placed on the segment 0–1 for maximum stress in L_0L_2; one-third is placed on the segment 0–2 for maximum stresses in U_1U_2 and L_2L_3; one-half is placed on segment 0–3 for maximum stress in U_2U_3.

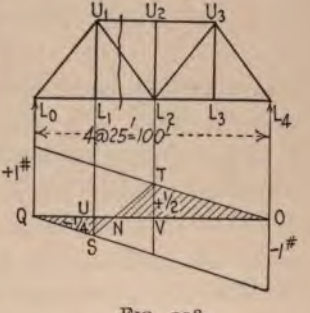


FIG. 108.

100. The stress in a web member of a parallel chord truss is usually determined by passing a section through the web member and two chord members, in such a manner that the truss is cut into two portions. The shear, or algebraic sum, of the vertical forces acting on one side (either side) of the section, equals the vertical component of the stress in the web member.

101. Influence Line for Shear in a Panel.—The criterion for the vertical component of the maximum stress in U_1L_2 (Fig. 108) will be developed by drawing an influence line, showing the variation of the shear in panel 1–2 as a load of 1 lb. moves across the span.

When the load of 1 lb. is between L_4 and L_2, the shear in the panel equals the left reaction; which increases from 0 to 1 lb. as the load moves from L_4 to L_0. Hence, the line OT is the influence line for shear in the panel, when the load of 1 lb. is between L_4 and L_2; and $TV = +\frac{1}{2}$. When the load of 1 lb.

is between L_0 and L_1, the shear in the panel is the right reaction taken negatively. Therefore QS is the influence line for shear in the panel, when the load of 1 lb. is between L_0 and L_1; and $US = -\frac{1}{4}$.

When the load of 1 lb. is in the panel, or between L_1 and L_2, there are two forces acting on either side of the section. On the left, there are the truss reaction at L_0 and the floor beam load at L_1; while on the right there are the truss reaction at L_4 and the floor-beam load at L_2. Let the load of 1 lb. be in the panel and at a distance x from L_2. Consider the forces on the left of the section, and let R represent the truss reaction at L_0, r the floor-beam load at L_1 and F the shear in the panel 1–2; then

$$R = \frac{50 + x}{100} \text{ and } r = \frac{x}{25}$$

$$F = R - r = \frac{50 - 3x}{100}$$

This equation, being of the first degree, may be represented by a straight line. When the load of 1 lb. is at L_2, $x = 0$, and $F = \frac{1}{2}$. When the load of 1 lb. is at L_1, $x = 25$, and $F = -\frac{1}{4}$. Therefore the line TS, having the ordinates $TV = \frac{1}{2}$ at L_2, and $US = -\frac{1}{4}$ at L_1, is the influence line for shear in the panel; when the load of 1 lb. is between L_1 and L_2. Hence, the line $OTNSQ$ is the influence line for shear in the panel 1–2.

Positive shear in panel 1–2 causes tension in U_1L_2. As the load of 1 lb. moves to the left from L_4, the tension in U_1L_2 increases to a maximum as the load arrives at L_2. As the load moves forward, the tension decreases rapidly, and the stress in U_1L_2 becomes zero when the load is directly over N. As the load passes N, the member resists a compressive stress, increasing to a maximum as the load reaches L_1; after which the compression decreases until the stress becomes zero, as the load of 1 lb. leaves the span.

102. Criterion for Maximum Shear in a Panel.—The shear in panel 1–2 (Fig. 109) may be determined by multiplying each load by its corresponding ordinate in the influence line diagram, and finding the sum of these products. The influence line is composed of three lines QS, ST and TO; and the loads over each

line may be combined respectively into P_1, and P_2 and P_3 as shown. Let y_1, y_2 and y_3 represent respectively the ordinates under the centers of gravity of P_1, P_2 and P_3; and let F represent the shear in the panel; then

$$F = P_3y_3 \pm P_2y_2 - P_1y_1$$

The ordinate y_2 will have a plus or minus sign as the resultant P_2 is on the right or left of N.

Now let the loads move a small distance d to the left, in such a way that no loads cross the panel points L_0, L_1, L_2 or L_4.

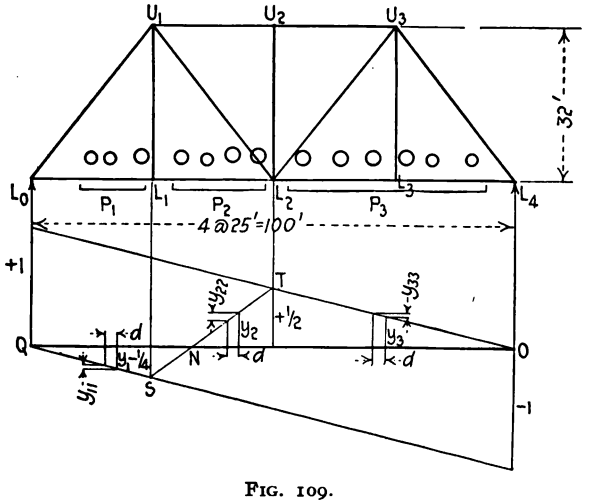


FIG. 109.

Or in other words, P_1, P_2 and P_3 are to remain constant.

The change in shear in the panel is

$$\Delta F = P_3y_{33} - P_2y_{22} + P_1y_{11}.$$

If y_2 had been a negative ordinate, it would have been *lengthened* by the movement, an amount y_{22}. In either case the shear is decreased and y_{22} is negative. Is this change in F, caused by a movement of the loads to the left, a positive or a negative quantity? The answer depends upon whether

$$P_2y_{22} \lessgtr P_3y_{33} + P_1y_{11}$$

for as long as $P_2y_{22} < P_3y_{33} + P_1y_{11}$

the change in F, caused by a movement of the loads to the *left*, will be a positive quantity.

From similar triangles

$$y_{11}:d::l:100$$
$$y_{22}:d::\tfrac{3}{4}:25$$
$$y_{33}:d::l:100$$

or $$y_{11} = \frac{d}{100},\ y_{22} = \frac{3d}{100} \text{ and } y_{33} = \frac{d}{100}.$$

Hence the change in F, caused by a movement of the loads to the *left*, will be a positive quantity as long as

$$\frac{3d}{100} P_2 < \frac{d}{100} P_3 + \frac{d}{100} P_1$$

or as long as $3P_2 < P_3 + P_1$.

Likewise the change in F, caused by a movement of the loads to the *right* will be a *negative* quantity so long as

$$3P_2 < P_3 + P_1.$$

Let $P_1 + P_2 + P_3 = P$ = total load on the span;

if $3P_2 < P_3 + P_1$

and $P_2 = P_2$

then $4P_2 < P$

or $$P_2 < \frac{P}{4}$$

Hence, a *positive* shear in panel 1–2 will be increased by a movement of the loads to the *left*, and a *negative* shear will be increased by a movement of the loads to the *right* so long as

$$P_2 < \frac{P}{4}$$

Hence, the criterion for maximum positive or negative shear in panel 1–2 is

$$P_2 \lessgtr \frac{P}{4}$$

For a maximum positive shear the influence line indicates that as many wheels as possible should be on the span between L_4 and L_2, and a few wheels (not exceeding one-fourth of the total load) should move across L_2 into the panel. For a maximum negative shear, as many wheels as possible should be on the span between L_0 and L_1, and a few loads (not exceeding

one-fourth of the total load) should move across L_1 into the panel.

Suppose that the panel lengths in Fig. 109 are changed so that $L_0L_1 = 20$ ft., $L_1L_2 = 25$ ft., $L_2L_3 = 38$ ft. and $L_3L_4 = 17$ ft. Will this change modify the criterion

$$P_2 \lessgtr \frac{P}{4}$$

and if so, how?

103. Illustrative Problems.

What is the maximum tension in U_1L_2 (Fig. 109) for an E-40 loading? The maximum compression?

Tension.—Move the train on the span from the right end until wheel 2 is at L_2. Wheel 10 is 2 ft. on the span and the total load is 152.

$$\begin{array}{r} 4)152 \\ 38 \gtrless 10 \\ \gtrless 30 \end{array} \quad \text{N.G.}$$

When wheel 2 is approaching, or has passed L_2, the load in the panel is less than one-fourth the total load on the span; consequently the shear in the panel is increasing in either case.

Try wheel 3 at L_2.

$$\begin{array}{r} 4)152 \\ 38 \gtrless 30 \\ < 50 \end{array} \quad \text{O.K.}$$

Wheel 3 satisfies the criterion for maximum positive shear in the panel, for the load in the panel is less than one-fourth the total load when wheel 3 is approaching L_2, and the shear is increasing; also the load in the panel is greater than one-fourth the total load when wheel 3 has passed L_2, and the shear is decreasing. Make a sketch of the truss showing the train with wheel 3 at L_2.

The shear in the panel when wheel 3 is at L_2 is determined by finding the algebraic sum of the vertical forces acting on one side (either side) of a section through the panel. There are four forces acting on the right, *i.e.*, the truss reaction at L_4 and the three floor beam loads at L_2, L_3 and L_4 respectively. There are only two forces acting on the left of the section, *i.e.*, the truss reaction at L_0 and the floor beam load at L_1. The shear in the panel will therefore be computed by considering

the forces acting on the left of the section. Wheel 10 is 7 ft. on the span.

$$\begin{aligned} 152 \times 7 &= 4{,}632 \\ &\quad 1{,}064 \\ 100)\,&5{,}696 \\ &56.96 = \text{left reaction} \\ \frac{230}{25} &= 9.2 = \text{floor-beam load at } L_1 \\ &47.76 = \text{shear in the panel.} \end{aligned}$$

The length of the member U_1L_2 is 40.6 ft.

$$47.76 \times \frac{40.6}{32} = 60.6 = \text{the maximum tension in } U_1L_2.$$

Compression.—Move the train on the span from the left until wheel 2 is at L_1. Wheel 6 is 1 ft. on the span and the total load is 103.

$$4)103$$
$$25.75 \begin{matrix} > 10 \\ < 30 \end{matrix} \text{ O.K.}$$

Wheel 2 satisfies the criterion for maximum negative shear in the panel, for the load in the panel is less than one-fourth the total load when wheel 2 is approaching L_1, and the negative shear is increasing; also the load in the panel is greater than one-fourth the total load when wheel 2 has passed L_1, and the negative shear is decreasing. Make a sketch of the truss showing the train with wheel 2 at L_1.

There are two forces acting on the right of the section *i.e.*, the truss reaction at L_4 and the floor beam load at L_2. Wheel 6 is 1 ft. on the span.

$$\begin{aligned} 103 \times 1 &= 1{,}640 \\ &\quad 103 \\ 100)\,&1{,}743 \\ &17.43 = \text{right reaction} \\ \frac{80}{25} &= 3.2 = \text{floor beam load at } L_2 \\ &14.23 = \text{negative shear in the panel} \end{aligned}$$

$$14.23 \times \frac{40.6}{32} = 18.1 = \text{the maximum compression in } U_1L_2.$$

The member U_1L_2 may have a maximum tension of 60.6,

or a maximum compression of 18.2. It is clear that, if the member U_1L_2 were removed and a member L_1U_2 substituted, the member L_1U_2 would have a maximum tension of 18.2 and a maximum compression of 60.6.

U_1L_1.—This member supports the floor beam load at L_1, consequently the stress is a maximum when the floor beam load is a maximum. As a load of 1 lb. moves from L_4 to L_2, the floor beam load is zero. The floor beam load increases uniformly from 0 to 1 lb. as the load moves from L_2 to L_1, and decreases uniformly from 1 lb. to 0 as the load moves from L_1 to L_0. The influence line diagram for the stress in U_1L_1 is

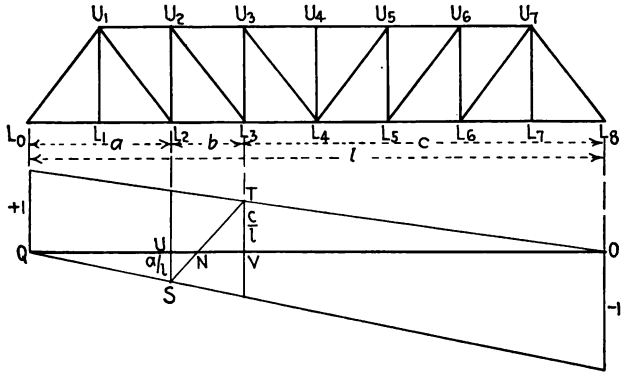


FIG. 110.

an isosceles triangle having a base 50 ft. long, and an altitude of 1 lb. under L_1. The criterion for maximum stress in U_1L_1, developed from this influence line, shows that the panels L_0L_2 should be loaded with one-half the load in each panel. Either wheel 4 or wheel 13 at L_1 satisfies this criterion; the stress being the same in either case. When wheel 4 is at L_1, wheel 8 is at L_2 and wheel 1 is 7 ft. from L_0. The stress in U_1L_1 which equals the floor beam load at L_1 is 75.6 tension.

104. General Criterion for Maximum Shear in a Panel.—Let F represent the shear in panel 2–3 of the parallel chord truss in Fig. 110, when the truss supports any system of vertical loads. Since F represents also the vertical component of the stress in U_2L_3, it is clear that the stress is a maximum

when the shear in the panel is a maximum. The influence for shear in panel 2–3 has been drawn as in Article **101.** If P_1 represents the load on the segment a; P_2, the load in the panel 1–2; P_3, the load on the segment c; and P the total load on the span, the criterion for maximum shear, which may be developed in the same manner as in Article **102,** is

$$P_2 \lessgtr \frac{b}{l} P$$

This criterion is not a function of either a or c, and consequently is applicable to any panel of the truss; when P_2 is the load in the panel, P is the total load on the span, b is the length of the panel and l is the length of the span. The criterion is often expressed in the form

$$P_2 = \frac{b}{l} P$$

or

$$\frac{P_2}{b} = \frac{P}{l}$$

and may be stated as follows: The maximum stress in any diagonal of a parallel chord truss (Pratt or Warren) occurs when the load in the panel divided by the panel length, equals the load on the span divided by the span length.

If the truss in Fig. 110 contains eight equal panels, it is clear that one-eighth of the total load on the span should be placed in any panel for maximum stress in the diagonal of that panel; the train approaches from the right for positive shear and from the left for negative shear.

In a through bridge, having the floor beams at the bottom chord, the maximum stress in a vertical member, U_2L_2 for example, equals the maximum shear in panel 2–3, for that portion of U_2L_2 between the top of the floor beam and U_2. The stress in that part of U_2L_2 (if any exists) between the bottom of the floor beam and the end connection differs from the stress above the floor beam by the amount of the floor beam load.

From similar triangles

$$\frac{c}{l} : \frac{a+c}{l} :: NV : b$$

therefore $NV : c :: b : a + c$

$NV : VO :: b : a + c$

$NV : NV + VO :: b : b + a + c$

or $\frac{NV}{NO} = \frac{b}{l}$

but since $\frac{P_2}{b} = \frac{P}{l}$

then $\frac{P_2}{NV} = \frac{P}{NO}$

Thus it is clear that, when the right portion of the span is loaded for maximum positive shear in any panel; the load in the panel divided by the length NV equals the total load in the span divided by the length NO. In other words, if the truss has n equal panels and the influence line $OTNSQ$ is drawn for shear in any panel

$$\frac{NV}{NO} = \frac{1}{n}$$

Similarly it may be shown that

$$\frac{NU}{NQ} = \frac{1}{n}$$

The influence line NTO, for positive shear in any panel, has the same general characteristics as an influence line for bending moment at the point V in a beam having the length NO; the criterions are the same and the positions of the train are the same in both cases. The position of the train for maximum negative shear, in any panel, is the same as for maximum bending moment at U in a beam having the length NQ.

105. Impact.—In computing live load stresses in a bridge, the train is considered as a static load, gently placed upon the structure in the required position for maximum stress in a given member. In order to provide for the increased stress caused by dynamic effect of the train in motion, an additional stress known as *impact* is combined with each live load stress. The amount of impact to be added is determined arbitrarily from an empirical formula. The one most frequently used is

$$I = L \frac{300}{300 + l}$$

in which

I = impact stress
L = live load stress
l = length of train causing the live load stress.

For example, the live load stress in U_1U_3 (Fig. 106), as determined in Article **98**, is 100.6. The impact stress is

$$100.6 \times \frac{300}{300 + 100} = 75.4 \text{ compressive}$$

The live load tension stress in U_1L_2 (Fig. 108), as determined

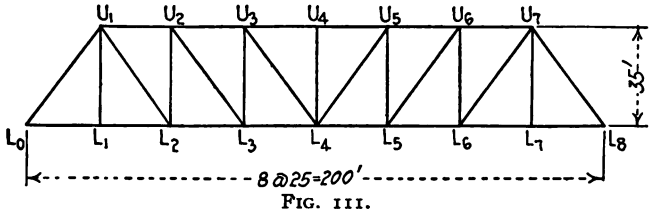


FIG. 111.

in Article **103** is 60.6. The impact stress is

$$60.6 \times \frac{300}{300 + 63} = 50.2 \text{ tensile}$$

The live load compression stress is 18.1, and the impact is

$$18.1 \times \frac{300}{300 + 33} = 16.3 \text{ compressive}$$

The live load stress in U_1L_1 is 75.6. The length of the train causing this stress extends over two panels only, hence the impact stress is

$$75.6 \times \frac{300}{300 + 43} = 66.1$$

106. Stresses in a 200-ft. Pratt Truss.—(Fig. 111). The assumed dead loads are as follows:

Track (rails and ties)	450
Floor (floor beams and stringers)	550
Two trusses and bracing	1,200
	2,200 lb. per linear foot

The live load is a Cooper's E-40 train. The impact will be taken in accordance with the formula of Article **105**.

Dead-load Stresses.—The dead load supported by each truss is 1,100 lb. per linear foot, or 27.5 per panel. The stresses

may be determined by placing a load of 27.5 at the seven joints L_1 to L_7 and drawing a stress diagram; but in the case of parallel chords the algebraic method is considerably shorter. One-half the load per linear foot on each truss is 550 lb. (see Article 59).

$$\frac{0.550 \times 1 \times 25 \times 7 \times 25}{35} = 9.82 \times 1 \times 7 = 68.7 = L_0L_2$$

$$9.82 \times 2 \times 6 = 117.8 = U_1U_2 = L_2L_3$$

$$9.82 \times 3 \times 5 = 143.7 = U_2U_3 = L_3L_4$$

$$9.82 \times 4 \times 4 = 157.1 = U_3U_4$$

The length of each diagonal is 43 ft. The shear in panel 3–4 is one-half a panel load, or 13.75

$$13.75 \times \frac{43}{35} = 16.9 = U_3L^4$$

$$16.9 \times 3 = 50.7 = U_2L_3$$

$$16.9 \times 5 = 84.5 = U_1L_2$$

$$16.9 \times 7 = 118.3 = L_0U_1$$

$$13.75 = U_3L_3$$

$$13.75 \times 3 = 41.3 = U_2L_2$$

$$27.5 = U_1L_1$$

$$0 = U_4L_4$$

It is more accurate to consider one-half the weight of the truss as concentrated at the top chord panel points. This would affect the stresses in the vertical members only, increasing the stress in U_2L_2, U_3L_3 and U_4L_4 and decreasing the stress in U_1L_1 by 7.5

Live load and Impact Stresses

L_0L_2 Wheel 4 at L_1

$$\begin{array}{r} 16{,}364 \\ 284 \times 84 = 23{,}856 \\ 84^2 = \underline{\ 7{,}056} \\ 8)\overline{47{,}276} \\ 5{,}909.5 \\ \underline{\quad 480.} \\ 35)\overline{5{,}429.5} \\ LL = 155.1 \end{array}$$

$$I = 155.1 \times \frac{300}{493} = 94.5$$

U_1U_2 and L_2L_3 Wheel 7 at L_2

$$\begin{aligned} & 16{,}364 \\ 284 \times 78 = & \ 22{,}152 \\ 78^2 = & \ 6{,}084 \\ & \overline{4)44{,}600} \\ & 11{,}150 \\ & \ 2{,}155 \\ & \overline{35)\ 8{,}995} \\ LL = & \ 257 \end{aligned}$$

$$I = 257 \times \frac{300}{487} = 158.2$$

U_2U_3 and L_3L_4 Wheel 11 at L_3

$$\begin{aligned} & 16{,}364 \\ 284 \times 80 = & \ 22{,}720 \\ 85^2 = & \ 6{,}400 \\ & \overline{45{,}484} \\ & \ 0.375 \\ & \overline{17{,}056.5} \\ & \ 5{,}848 \\ & \overline{35)11{,}208.5} \\ LL = & \ 320.2 \end{aligned}$$

$$I = 320.2 \times \frac{300}{489} = 196.5$$

U_3U_4 Wheel 13 at L_4

$$\begin{aligned} & 16{,}364 \\ 284 \times 65 = & \ 18{,}460 \\ 65^2 = & \ 4{,}225 \\ & \overline{2)39{,}049} \\ & 19{,}524.5 \\ & \ 7{,}668 \\ & \overline{35)11{,}856.5} \\ LL = & \ 338.8 \end{aligned}$$

$$I = 338.8 \times \frac{300}{474} = 214.5$$

The stresses in all chord members of the left half of the truss are a maximum for positions of the train when crossing the span from right to left. This is not a general statement, to be accepted for all trusses, although it is true for this truss. The chord members of the right half will have their maximum stresses when the train crosses from left to right. The stresses for all web members will be computed for the train moving from right to left. The length of any diagonal member is 43 ft. Hence if

v = the vertical component of the stress in a diagonal
and s = the stress in the diagonal

then $s = \frac{v}{35} \times 43$

or $s = 1.23v$

L_6U_7 Wheel 2 at L_7

$$\begin{array}{r} 1{,}640 \\ 103 \times 1 = \underline{\quad 103} \\ 200)\overline{1{,}743} \\ \quad 8.7 \end{array}$$

$$\frac{80}{25} = \frac{3.2}{5.5} \times 1.23 = 6.8 = LL$$

$$6.8 \times \frac{300}{333} = 6.1 = I$$

U_6L_6 and L_5U_6 Wheel 2 at L_6

$$\begin{array}{r} 4{,}632 \\ 152 \times 2 = \underline{\quad 304} \\ 200)\overline{4{,}936} \\ \quad 24.7 \\ \quad \underline{3.2} \\ \quad 21.5 = LL \\ 21.5 \times \frac{300}{358} = 18. = I \end{array} \Bigg\} U_6L_6$$

$$\left.\begin{array}{l} 21.5 \times 1.23 = 26.5 = LL \\ 18 \times 1.23 = 22.1 = I \end{array}\right\} L_5U_6$$

U_5L_5 and L_4U_5 Wheel 2 at L_5

$$\begin{array}{r} 8{,}728 \\ 232 \times 4 = \underline{\quad 928} \\ 200)\overline{9{,}656} \\ 48.3 \\ \underline{\quad 3.2} \\ 45.1 = LL \\ 45.1 \times \frac{300}{383} = 35.3 = I \end{array} \Big\} U_5L_5$$

$$\left.\begin{array}{l} 45.1 \times 1.23 = 55.4 = LL \\ 35.3 \times 1.23 = 43.4 = I \end{array}\right\} L_4U_5$$

U_3L_3 and U_3L_4 Wheel 3 at L_4

$$\begin{array}{r} 16{,}364 \\ 284 \times 4 = \quad 1{,}136 \\ 4^2 = \qquad 16 \\ 200)\overline{17{,}516} \\ 87.6 \\ \frac{230}{25} = \underline{\quad 9.2} \\ 78.4 = LL \\ 78.4 \times \frac{300}{413} = 57. = I \end{array} \Big\} U_3L_3$$

$$\left.\begin{array}{l} 78.4 \times 1.23 = 96.4 = LL \\ 57. \times 1.23 = 70.1 = I \end{array}\right\} U_3L_4$$

U_2L_2 and U_2L_3 Wheel 3 at L_3

$$\begin{array}{r} 16{,}364 \\ 284 \times 29 = 8{,}236 \\ 29^2 = \quad 841 \\ 200)\overline{25{,}441} \\ 127.2 \\ \underline{\quad 9.2} \\ 118. = LL \\ 118 \times \frac{300}{438} = 80.9 = I \end{array} \Big\} U_2L_2$$

$$\left.\begin{array}{l} 118. \times 1.23 = 145.2 = LL \\ 80.9 \times 1.23 = \quad 99.6 = I \end{array}\right\} U_2L_3$$

U_1L_2 Wheel 3 at L_2

$$\begin{aligned} &16{,}364 \\ 284 \times 54 &= 15{,}336 \\ 54^2 &= \underline{\ 2{,}916} \\ 200)&34{,}616 \\ &173.1 \\ &\underline{\ 9.2} \\ &163.9 \times 1.23 = 201.6 = LL \\ &201.6 \times \frac{300}{463} = 130.6 = I \end{aligned}$$

L_0U_1 Wheel 4 at L_1

$$\begin{aligned} &16{,}364 \\ 284 \times 84 &= 23{,}856 \\ 84^2 &= \underline{\ 7{,}056} \\ 200)&47{,}276 \\ &236.4 \\ \frac{480}{25} &= \underline{19.2} \\ &217.2 \times 1.23 = 267.2 = LL \\ &267.2 \times \frac{300}{493} = 162.5 = I \end{aligned}$$

U_1L_1 Wheel 4 at L_1

Moment of wheels 1 to 8 about 8 = 2851
Moment of wheels 1 to 4 about 4 = 480

$$\begin{aligned} &2{,}851 \\ 480 \times 2 &= \underline{960} \\ 25)&1{,}891 \\ &75.6 = LL \\ 75.6 \times \frac{300}{343} &= 66.1 = I \end{aligned}$$

When the train crosses the span in one direction, it is obvious that the stress in any web member of the left half is the same as the stress in the corresponding member of the right half, when the train crosses in the opposite direction. Considering one-half of the structure, we shall have the various combinations of

stresses, as listed in Fig. 112. When the dead and live load stresses are opposite in character, only two-thirds of the dead-load stress is considered. Thus in U_3L_4 the total stress is $+183.4$ when the train crosses from right to left, and -87.5 when the train crosses from left to right.

In U_1L_2 the dead-load tensile stress is greater than the live-

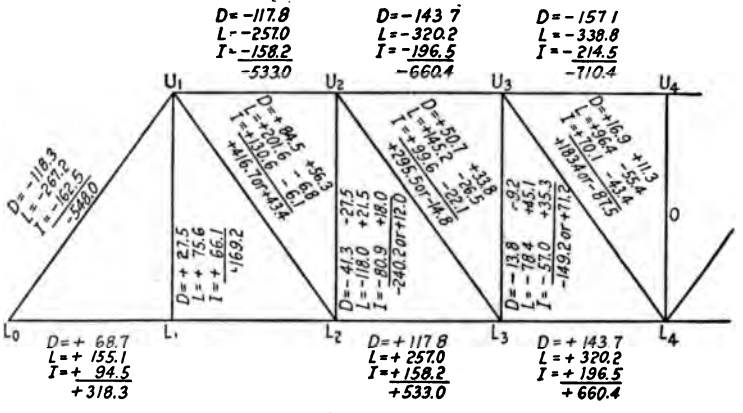


FIG. 112.

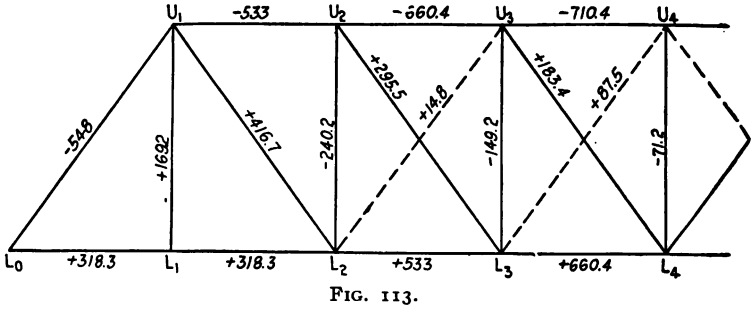


FIG. 113.

load and impact compressive stresses, and there is no reversal.

107. Counters.—The diagonals in two of the panels in Fig. 112 must be designed to resist both tensile and compressive stresses. If eye-bars or other flexible members, which are not capable of taking compressive stress, are used; it will be necessary to introduce counters in panels 2–3 and 3–4, as shown by the dotted lines in Fig. 113.

A counter carries no stress, except when the shear in the panel is of such a nature that the stress in the main diagonal would be compressive if the counter were omitted. The tensile stress carried by the counter in a parallel chord truss, obviously has the same magnitude as the compressive stress in the main diagonal if no counter were used. Thus in Fig. 113 the maximum stress in U_3L_4 is $+183.4$, and the maximum stress in L_3U_4 is $+87.5$. The maximum compressive stress in U_4L_4 has the same magnitude as the maximum tensile stress in U_3L_3, when no counters are used. When the bridge is loaded for maximum chord stresses, the counters are not in action and consequently have no effect upon the maximum chord stresses.

The Pratt truss in Fig. 112 may be transformed into a Warren truss, by substituting the diagonal L_2U_3 for the diagonal U_2L_3. The stress in L_2U_3 will be -295.5 or $+14.8$; the stress in U_2U_3 will be the same as in U_1U_2; the stress in L_2L_3 will be the same as in L_3L_4; the stress in U_2L_2 will be the same as in U_4L_4; and the stress in U_3L_3 will be the same as in U_1L_1.

108. Truss with an Odd Number of Panels.—The bottom chord in the center panel of a truss having an odd number of panels requires a special consideration. The truss in Fig. 114 is loaded for the maximum moment at L_4. The moment at L_4 is 9,088.57 ft.-lb., and the moment at L_3 is 8,929.43 ft.-lb. Since the moment at L_4 is greater than the moment at L_3, the shear in the panel 3–4 is positive, and the diagonal U_3L_4 is acting. The shear in panel 3–4 equals the left reaction, minus the loads from wheel 1 to wheel 9 inclusive, minus that part of the load in the panel which is carried at L_3, or

$$F = 167.57 - 142 - 19.2 = 6.37$$

The shear may also be found by dividing the difference in moments at L_4 and L_3 by the panel length, thus

$$F = \frac{9{,}088.57 - 8{,}929.43}{25} = 6.37$$

The truss in Fig. 115 is loaded for the maximum moment at L_3. The moment at L_3 is 9,155.86 ft.-lb., and the moment at L_4 is 8,944.14 ft.-lb. Since the moment at L_3 is greater than

the moment at L_4, the shear in the panel 3–4 is negative and the diagonal L_3U_4 is acting. The shear in panel 3–4 is

$$F = 200.05 - 172 - 36.52 = -8.47$$

or

$$F = \frac{8{,}944.14 - 9{,}155.86}{25} = -8.47$$

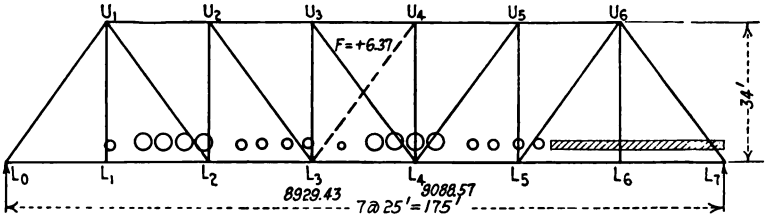


FIG. 114.

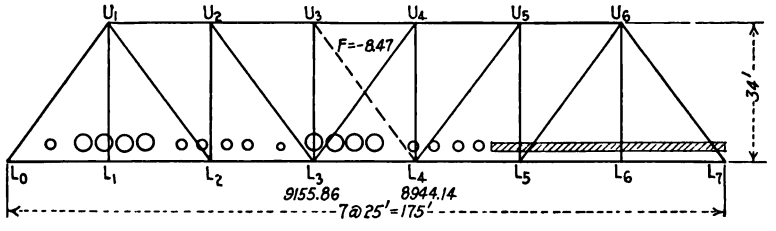


FIG. 115.

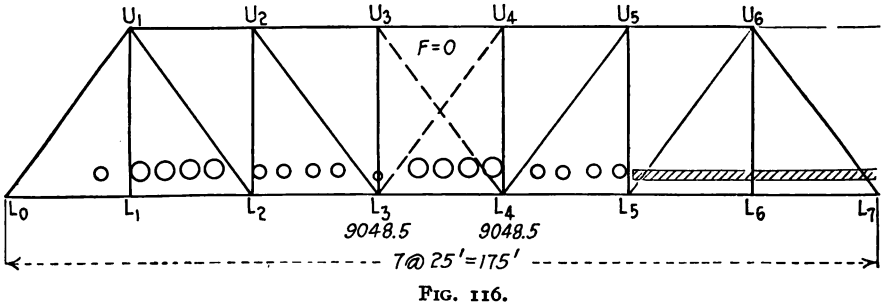


FIG. 116.

The maximum stress in U_3U_4 is $9{,}155.86/34 = -269.3$. This stress occurs when the train crosses the span from right to left with wheel 11 at L_3; or when the train crosses from left to right and wheel 11 is at L_4. Hence the maximum stress in U_2U_3, U_3U_4 and U_4U_5 is -269.3.

The stress in L_3L_4 is obtained by dividing the moment at U_3 in Fig. 114, or the moment at U_4 in Fig. 115, by 34; but in either case the moment at the other end of the panel is greater. It is clear, therefore, that the maximum stress in L_3L_4 occurs when the moments at L_3 and L_4 are equal, or when the shear in the panel is zero and neither diagonal is acting.

The total load on the span in Fig. 114 is 364.

$$\frac{364}{7} = 52\lesseqgtr{50}{70}$$

The load in panel 3–4 is less or greater than one-seventh of the load on the span as wheel 13 is crossing L_4, therefore the shear in the panel is a maximum (not *the* maximum to be sure, but still a maximum). There are three intermediate critical positions of the train between those shown in Figs. 114 and 115—wheel 14 at L_4, wheel 10 at L_3 and wheel 15 at L_4—and for each position the shear in panel 3–4 is decreasing, for the load in the panel in each case is greater than one-seventh of the load on the span. Hence, as the train moves from the position in Fig. 114 to the position in Fig. 115, and the shear in panel 3–4 decreases from +6.37 to −8.47; there is one, and only one, position of the train for which the shear in the panel is zero.

In the present case the shear in panel 3–4 is (approximately) zero when wheel 10 is at L_3 (Fig. 116) and the moment at either U_3 or U_4 is 9,048.5 ft.-lb. Hence, the maximum stress in L_3L_4 is +266.1. This stress is slightly less than the stress in U_3U_4, and in practice is usually assumed the same as the stress in U_3U_4.

SEC. IV. PARKER TRUSSES

109. Stress in Web Member of Parker Truss.—It was shown in Article 99 that the criterion for maximum stress in any chord member of the Parker truss (Fig. 107) is the same as for the corresponding member of the Pratt truss. It is true also that the criterion for maximum shear in any panel of the Parker truss is the same as for the corresponding panel in the Pratt truss. In a consideration of the maximum stresses for web members, a clear distinction should be made. In the

Pratt truss, the stress in any diagonal is proportional to the shear in the panel; and therefore is a maximum when the shear in the panel is a maximum. In the Parker truss, the stress in any diagonal U_1L_2, for example, is *not* proportional to the shear in the panel, for the vertical component of the stress in U_1U_2 enters into the solution.

110. Influence Line for Web Member of Parker Truss.—If

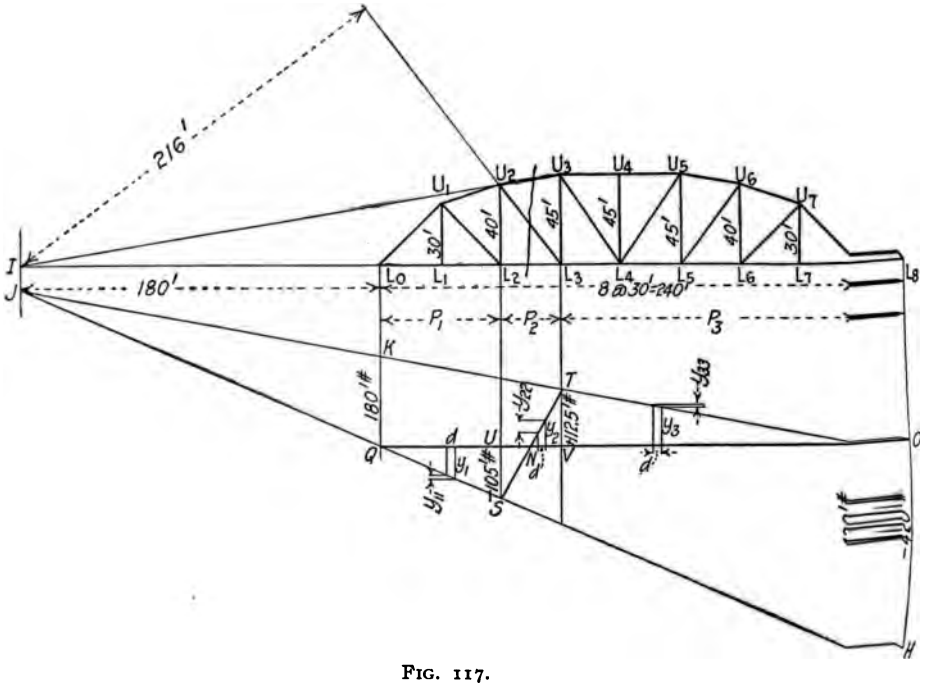


FIG. 117.

the truss in Fig. 117 is loaded in any manner, the stress in U_2L_3 may be determined by passing a section through the panel 2–3, and balancing the moments of all the forces acting on one side (either side) of the section about the intersection of the two chord members cut by the section. Let M_I represent the algebraic sum of the moments about I of all the forces acting on one side (either side) of the section through the panel 2–3, as a load of 1 lb. moves across the span.

When the load of 1 lb. is between L_8 and L_3, the only force

acting on the left of the section is the left reaction R_0, and $M_1 = 180R_0$. As the load moves from L_8 to L_0, R_0 increases uniformly from 0 to 1 lb., and $180R_0$ increases uniformly from 0 to 180 ft.-lb. Likewise the ordinate to the line OK, increasing uniformly from zero at O to 180 ft.-lb. at Q, represents $180R_0$. Hence, the line OT is the influence line for $M_I = 180R_0$, when the load of 1 lb. is between L_8 and L_3. M_I for the force acting on the left of the section is counter-clockwise about I, and for this reason QK is laid off above the line OQ.

When the load of 1 lb. is between L_0 and L_2, the only force acting on the right of the section is the right reaction R_8 and $M_I = 420R_8$. As the load moves from L_0 to L_8, R_8 increases uniformly from 0 to 1 lb. and $420R_8$ increases uniformly from 0 to 420 ft.-lb. Likewise the ordinate to the line QH, increasing uniformly from zero at Q to 420 ft.-lb. at O, represents $420R_8$. Hence the line QS is the influence line for M_1, when the load of 1 lb. is between L_0 and L_2. M_I for the force acting on the right of the section is counter-clockwise, consequently M_I for the forces acting on the left of the section is clockwise, and for this reason OH is laid off below the line OQ.

$$TV = 180 \times \frac{5}{8} = 112.5$$

$$US = -420 \times \frac{2}{8} = -105$$

When the load of 1 lb. is between L_2 and L_3, there are two forces acting on either side of the section. On the left, there are the truss reaction R_0 at L_0, and the floor-beam load r_2 at L_2; on the right of the section, there are the truss reaction R_8 at L_8, and the floor-beam load r_3 at L_3. Let the load of 1 lb. be in the panel 2–3 at a distance x from L_3, and consider the forces acting on the left of the section.

$$R_0 = \frac{150 + x}{240}, \quad r_2 = \frac{x}{30}$$

$$M_I = 180R_0 - 240r_2$$

$$M_1 = \frac{180(150 + x)}{240} - \frac{240x}{30}$$

$$M_I = \frac{450 - 29x}{4}$$

This equation, being of the first degree, may be represented by a straight line. When the load of 1 lb. is at L_3, $x = 0$ and $M_1 = 112.5$. When the load of 1 lb. is at L_2, $x = 30$, and $M_I = -105$. Therefore, TS, having the ordinates $TV = 112.5$ at L_3, and $US = -105$ at L_2, is the influence line for M_1 when the load of 1 lb. is in the panel 2–3. The stress in U_2L_3 is $M_I \div 216$. The stress caused by the load of 1 lb. in any position, may be found by dividing the corresponding ordinate in the influence line $OTNSQ$ by 216. As the load of 1 lb. moves to the left from L_8, the tension in U_2L_3 increases to a maximum, as the load arrives at L_3. As the load moves forward, the tension decreases rapidly and the stress in U_2L_3 becomes zero, when the load arrives at the point directly above N. As the load passes N, the member resists a compressive stress increasing to a maximum as the load arrives at L_2; after which the compression decreases to zero, as the load leaves the span at L_0.

Since $$IL_0 : IL_8 :: KQ : OH$$

it is evident that the two lines OK and HQ produced, intersect at J on the vertical through I. This influence resembles somewhat the influence line for shear in the panel 2–3. If the length of either member U_2L_2 or U_3L_3 were changed so that U_2U_3 approached a direction parallel to L_2L_3, the distance between I and L_0 would approach infinity, and the lines KO and QH would approach a condition of being parallel.

111. Criterion for Maximum Stress in Web Member of Parker Truss.—The member U_2L_3 (Fig. 117) will be used as an example. A train is assumed to be on the span. The stress in U_2L_3 is $M_I \div 216$. The value of M_I may be determined by taking the sum of the products of each load and its corresponding ordinate in the influence line. The loads will be combined into three groups P_1, P_2 and P_3, corresponding to the three portions of the influence line QS, ST and TO. If y_1, y_2 and y_3 represent respectively the ordinates under the centers of gravity of P_1, P_2 and P_3, then

$$M_I = P_3y_3 \pm P_2y_2 - P_1y_1.$$

Let the loads move a small distance d to the left in such a

way that no loads cross the panel points L_0, L_2, L_3 and L_8. Or in other words P_1, P_2 and P_3 are to remain constant. Is this change in M_I, caused by a movement of the loads to the left, a positive or a negative quantity? The answer depends upon whether

$$P_2y_{22} \lessgtr P_3y_{33} + P_1y_{11}$$

for as long as $P_2y_{22} < P_3y_{33} + P_1y_{11}$

the change in M_I caused by a movement of the loads to the left, will be a positive quantity.

From similar triangles

$$y_{11}:d::420:240$$
$$y_{22}:d::217.5:30$$
$$y_{33}:d::180:240$$

or $$y_{11} = \frac{7d}{4},\ y_{22} = \frac{29d}{4} \text{ and } y_{33} = \frac{3d}{4}$$

Hence the change in M_I, caused by a movement of the loads to the *left*, will be a *positive* quantity as long as

$$\frac{29d}{4}P_2 < \frac{3d}{4}P_3 + \frac{7d}{4}P_1$$

or as long as $29P_2 < 3P_3 + 7P_1$

Likewise, the change in M_I, caused by a movement of the loads to the *right*, will be a *negative* quantity as long as

$$29P_2 < 3P_3 + 7P_1.$$

The tensile stress in U_2L_3 is a maximum, when M_I has a maximum positive value, or when the right portion of the span is loaded. As the loads come onto the span at L_8 and move to the left, the tension in U_2L_3 increases as long as

$$29P_2 < 3P_3 + 7P_1.$$

Let $P_1 + P_2 + P_3 = P$ the total load on the span.

If $29P_2 < 3P_3 + 7P_1$

and $3P_2 = 3P_2$

then $32P_2 < 3P + 4P_1$

or $$P_2 < \frac{3}{32}P + \frac{1}{8}P_1.$$

Hence the criterion for maximum tensile stress in U_2L_3 is

$$P_2 \lessgtr \frac{3}{32}P + \frac{1}{8}P_1.$$

This criterion will be satisfied when the critical load is at L_2 and a few loads are in panel 2–3. In the case of all ordinary train loads the criterion will be satisfied before any loads pass L_2; in which case $P_1 = 0$ and the criterion for maximum tensile stress in U_2L_3 becomes

$$P_2 \lessgtr \frac{3}{32} P$$

The compressive stress in U_2L_3 is a maximum when M_I has a maximum negative value, or when the left portion of the span is loaded. As the loads come onto the span at L_0 and move to the *right* the compression in U_2L_3 increases as long as

$$29P_2 < 3P_3 + 7P_1$$

If $$29P_2 < 3P_3 + 7P_1$$

$$7P_2 = 7P_2$$

then $$36P_2 < 7P - 4P_3$$

or $$P_2 < \frac{7}{36} P - \frac{1}{9} P_3$$

Hence the criterion for maximum compressive stress in U_2L_3 is

$$P_2 \lessgtr \frac{7}{36} P - \frac{1}{9} P_3$$

This criterion will be satisfied when the critical load is at L_2, and a few loads are in panel 2–3. In the case of ordinary train loads the criterion will be satisfied before any loads pass L_3; in which case $P_3 = 0$ and the criterion becomes

$$P_2 \lessgtr \frac{7}{36} P.$$

112. Illustrative Problems.

1. What is the maximum tensile stress in U_2L_3 (Fig. 117) for an E-40 loading?

The train should move on to the span at L_3 and move to the left as long as

$$P_2 < \frac{3}{32} P$$

Try wheel 3 at L_3.

$$\begin{array}{r} 13 \\ 150 \\ \hline 163 \\ 109 \\ \hline 54 \\ 2 \\ \hline 108 \\ 284 \\ \hline 392 \end{array} = \text{total load on span}$$

$$392 \times \frac{3}{32} = 37 \gtrless^{30}_{50} \text{ O.K.}$$

Wheel 3 at L_3 satisfies the criterion, for when wheel 3 is approaching L_3 the load in the panel is less than $\frac{3}{32}$ of the total load on the span and when wheel 3 has passed L_3 the load in the panel is greater than $\frac{3}{32}$ of the total load on the span.

$$\begin{array}{r} 16{,}364 \\ 284 \times 54 = 15{,}336 \\ 54^2 = \underline{2{,}916} \\ 240)\overline{34{,}616} \\ 144{\cdot}23 = \text{left reaction} \end{array}$$

$$\frac{230}{30} = 7.67 = \text{floor-beam load at } L_2$$

$$\begin{array}{r} 144.23 \times 180 = 25{,}962 \\ 7.67 \times 240 = \underline{1{,}840} \\ 216)\overline{24{,}122} = M_1 \\ 111.7 = \text{maximum tension in } U_2L_3. \end{array}$$

2. What is the maximum compressive stress in U_2L_3? The train should move on to the span at L_0 and move to the right so long as

$$P_2 < \frac{7}{36}P$$

Try wheel 2 at L_2.

Wheel 11 is 4 ft. on the span and the total load is 172.

$$172 \times \frac{7}{36} = 33.4 \gtrless^{10}_{30} \text{ N.G.}$$

Try wheel 3 at L_2.

Wheel 12 is 4 ft. on the span and the total load is 192.

$$192 \times \frac{7}{36} = 37.3 > \frac{30}{50} \text{ O.K.}$$

$$\begin{array}{r} 6{,}708 \\ 768 \\ \hline 240)7{,}476 \\ 31.15 = \text{right reaction} \end{array}$$

$$\frac{230}{30} = 7.67 = \text{floor-beam load at } L_3$$

$$\begin{array}{r} 31.15 \times 420 = 13{,}083 \\ 7.67 \times 270 = 2{,}070 \\ \hline 216)11{,}013 = M_1 \\ 51 = \text{maximum compression in } U_2L_3. \end{array}$$

If the member U_2L_3 is made of eye-bars which are not capable of taking compression, it may be found necessary to introduce a counter L_2U_3 in the panel. The lever arm from the point I to this member is 199.7, hence, the maximum live load tensile stress which this counter would carry is

$$\frac{11{,}013}{199.7} = 55.2$$

Since the chord members U_3U_4 and L_3L_4 are parallel, the position of the train for maximum stress in U_3L_4 is obtained from the criterion for shear in panel 3–4; *i.e.*, one-eighth of the load on the span is placed in the panel whether loading from the right for tension or from the left for compression.

It has been shown that the criterion for maximum tension in U_2L_3 is not the same as the criterion for maximum compression. In this respect, members U_1L_2, and U_2L_2 are in the same category as U_2L_3. The criterions are not the same for any two members, neither are the criterions for maximum tension and compression the same for any one member. The center of moments for U_2L_3 and U_3L_3 is at the same point, but the section is not the same for both members. The center of moments for U_1L_2 and U_2L_2 is not at the same point as for U_2L_3 and U_3L_3.

113. Problems.

Draw the influence line, develop the criterion, and compute

the maximum tensile and compressive stresses in U_1L_1, U_1L_2, U_2L_2, U_3L_3 and U_3L_4 (Fig. 117), using an E-40 train. Compute the impact stress in each case, using the formula of Article **105.**

114. General Criterion for Maximum Stress in Web Member of a Parker Truss.—Let M_1 represent the algebraic sum of the moments of all the forces acting on one side (either side) of the section through the panel 2–3 (Fig. 118). Since the stress in

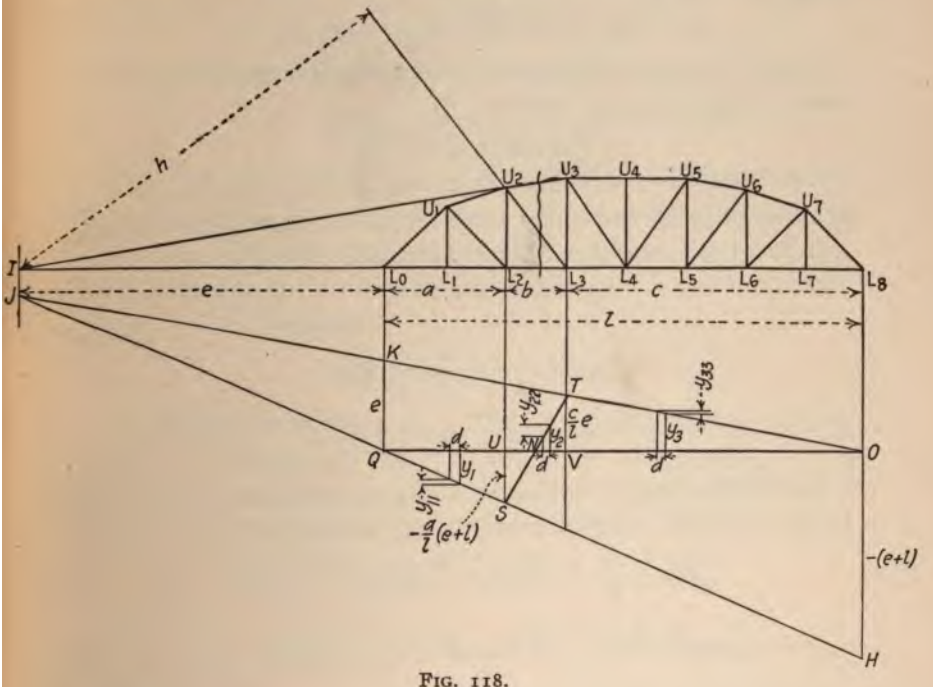


FIG. 118.

U_2L_3 equals $M_I \div h$, it is clear that the stress is a maximum when M_I is a maximum. Let a equal the length of the segment L_0L_2, b equal the length of the panel, c equal the length of the segment L_3L_8, and e equal the length IL_0; then $a + b + c = l$. The influence line $OTNSQ$ for M_I is drawn in the same manner as described in Article **110.** The value of M_I may be determined by taking the sum of the products of each load and its corresponding ordinate in the influence line. The loads will be combined into three groups, P_1, P_2 and P_3, corresponding

to the three portions of the influence line QS, ST and TO. If y_1, y_2 and y_3 represent respectively the three ordinates under the centers of gravity of P_1, P_2 and P_3, then

$$M_I = P_3y_3 \pm P_2y_2 - P_1y_1$$

Let the loads move a small distance d to the left in such a way that no loads cross the panel points L_0, L_2, L_3 and L_8. Or, in other words, P_1, P_2 and P_3 are to remain constant. The change in M_I is

$$\Delta M_I = P_3y_{33} - P_2y_{22} + P_1y_{11}$$

Is this change in M_I a positive or a negative quantity? The answer depends upon whether

$$P_2y_{22} \gtrless P_3y_{33} + P_1y_{11}$$

for as long as $P_2y_{22} < P_3y_{33} + P_1y_{11}$

the change in M_I caused by a movement of the loads to the left, will be a positive quantity. From similar triangles

$$y_{11}:d::e+l:l$$
$$y_{33}:d::e:l$$
$$y_{22}:d::\frac{c}{l}e + \frac{a}{l}(e+l):b$$

or $$y_{11} = \frac{d(e+l)}{l},\ y_{33} = d\frac{e}{l} \text{ and } y_{22} = \frac{d(ce+ae+al)}{bl}$$

Hence the change in M_I, caused by a movement of the loads to the left, will be a positive quantity as long as

$$\frac{d(ce+ae+al)}{bl}P_2 < \frac{de}{l}P_3 + \frac{d(e+l)}{l}P_1$$

or as long as $$\frac{ce+ae+al}{b}P_2 < eP_3 + (e+l)P_1$$

Likewise, the change in M_I, caused by a movement of the loads to the right, will be a negative quantity as long as

$$\frac{ce+ae+al}{b}P_2 < eP_3 + (e+l)P_1$$

The tensile stress in U_2L_3 is a maximum when M_I has a maximum positive value, or when the right portion of the span is loaded. As the loads come onto the span at L_8 and move to the left, the tension in U_2L_3 increases as long as

$$\frac{ce+ae+al}{b}P_2 < eP_3 + (e+l)P_1$$

Let $P_1 + P_2 + P_3 = P =$ the total load on the span.

If $$\frac{ce + ae + al}{b} P_2 < eP_3 + (e + l)P_1$$

and $$eP_2 = eP_2$$

then $$\frac{ce + ae + al + be}{b} P_2 < eP + lP_1$$

or $$\frac{(e + a)l}{b} P_2 < eP + lP_1$$

whence $$P_2 < (eP + lP_1) \frac{b}{(e + a)l}$$

Hence the criterion for maximum tensile stress in $U_2 L_3$ is

$$P_2 \lessgtr (eP + lP_1) \frac{b}{(e + a)l}$$

This criterion will be satisfied when the critical load is at L_3, and a few loads are in panel 2–3. In the case of all ordinary train loads, this criterion will be satisfied before any loads pass L_2, in which case $P_1 = 0$ and the criterion for maximum tensile stress in U_2L_3 becomes

$$P_2 \lessgtr \left(\frac{b}{l}\right)\left(\frac{e}{e + a}\right)P \qquad (1)$$

The compressive stress in U_2L_3 is a maximum when M_1 has a maximum negative value, or when the left portion of the span is loaded. As the loads come onto the span at L_0 and move to the right, the compression in U_2L_3 increases as long as

$$\frac{ce + ae + al}{b} P_2 < eP_3 + (e + l)P_1$$

If $$\frac{ce + ae + al}{b} P_2 < eP_3 + (e + l)P_1$$

and $$(e + l)P_2 = (e + l)P_2$$

then $$\frac{ce + ae + al + be + bl}{b} P_2 < (e + l)P - lP_3$$

or $$\frac{(a + b + e)l}{b} P_2 < (e + l)P - lP_3$$

whence $$P_2 < [(e + l)P - lP_3] \frac{b}{(a + b + e)l}$$

Hence the criterion for maximum compressive stress in U_2L_3 is

$$P_2 \lessgtr [(e + l)P - lP_3] \frac{b}{(a + b + e)l}$$

This criterion will be satisfied when the critical load is at L_2, and a few loads are in panel 2–3. In the case of all ordinary train loads, this condition will be satisfied before any load has passed L_3; in which case $P_3 = 0$ and the criterion for maximum compressive stress in U_2L_3 becomes

$$P_2 \lessgtr \left(\frac{b}{l}\right)\left(\frac{e+l}{e+a+b}\right)P \qquad (2)$$

Criterions (1) and (2) are easily remembered if the following observation is made. In computing the maximum tension in U_2L_3, M_I is determined from the truss reaction R_0 at L_0, and the floor-beam load r_2 at L_2, thus

$$M_I = eR_0 - (e+a)r_2 \qquad (3)$$

If the two chord members in panel 2–3 were parallel, this criterion for maximum tension in U_2L_3 would be

$$P_2 \lessgtr \frac{b}{l}P \qquad (4)$$

Criterion (4) may be transformed into criterion (1) by inserting the fraction $\frac{e}{e+a}$ in the coefficient of P. The numerator and denominator of this fraction appear as multipliers in Eq. (3).

Likewise in computing the maximum compression in U_2L_3, M_I is determined from the truss reaction R_8 at L_8, and the floor-beam load r_3 at L_3 thus

$$M_I = (e+l)R_8 - (e+a+b)r_3 \qquad (5)$$

If the two chord members in panel 2–3 were parallel, the criterion for maximum compression in U_2L_3 would be criterion (4); which may be transformed into criterion (2) by inserting the fraction $\frac{e+l}{e+a+b}$ in the coefficient of P. The numerator and denominator of this fraction appear as multipliers in Eq. (5).

Suppose (in Fig. 118) there are eight equal panels. For maximum stress in U_2L_3, the proportion of total load on the span which should be placed in the panel is $\frac{1}{8}$ times a fraction. The numerator of the fraction is the distance from the center

of moments to the truss reaction, and the denominator is the distance from the center of moments to the floor-beam load.

Thus in Fig. 117 the criterion for maximum tension in U_2L_3 is

$$P_2 < \frac{1}{8} \times \frac{180}{180 + 60} P$$

or

$$P_2 < \frac{3}{32} P$$

as given on page 166. The criterion for maximum compression in U_2L_3 is

$$P_2 < \frac{1}{8} \times \frac{180 + 240}{180 + 90} P$$

or

$$P_2 < \frac{7}{36} P$$

as given on page 167.

From similar triangles (Fig. 118)

$$\frac{ce}{l} : \frac{ce}{l} + \frac{a}{l}(e + l) :: NV : b$$

therefore

$$NV : c :: b : a + c + \frac{al}{e}$$

or

$$NV : VO :: b : a + c + \frac{al}{e}$$

whence

$$NV : NV + VO :: b : a + b + c + \frac{al}{e}$$

or

$$\frac{NV}{NO} = \frac{b}{l} \frac{e}{e + a}$$

substituting in criterion (1)

$$P_2 < \frac{NV}{NO} P$$

If the criterion is expressed as an equation, then

$$\frac{P_2}{NV} = \frac{P}{NO}$$

Thus it is clear that when the right portion of the truss is loaded for maximum tensile stress in U_2L_3, the load in the panel divided by the length NV equals the total load on the span divided by the length NO. Likewise it may be shown that for a maximum compressive stress in U_2L_3, the left portion

of the truss is loaded so that the load in the panel divided by NU equals the total load on the span divided by NQ.

Compute the lengths NV and NU in Fig. 117 and show that the criterions developed therefrom are the same as previously given.

115. Tension in a Vertical when Counters are Used.—The maximum tensile stress in U_2L_2 (Fig. 117) for an E-40 train with wheel 4 at L_2 is 68.7. The conditions are different if the

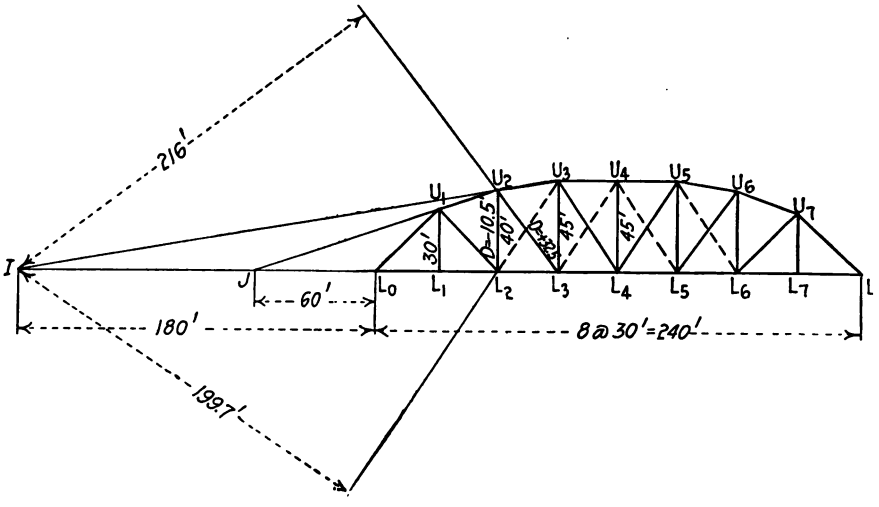


FIG. 119.

diagonal web members are designed to take tensile stresses only, as in Fig. 119, and the use of counters becomes necessary. The dimensions of the truss are the same as in Fig. 117. We shall assume that total dead load for the bridge is 2,600 lb. per linear foot, or 1,300 lb. to be carried by each truss; of which 950 lb. will be considered as acting at the bottom chord and 350 lb. at the top chord. The panel loads are 28,500 lb. for the bottom chord and 10,500 lb. for the top chord. The resulting dead load stresses for U_2L_2 and U_2L_3 are shown in the figure.

Let the train advance on the span at L_0 until wheel 7 is at L_0. The live load compressive stress in U_2L_3 is

$$\frac{2,155}{240} \times \frac{420}{216} = 17.45$$

and the impact compressive stress is

$$17.45 \times \frac{300}{337} = 15.55$$

The live-load and impact compressive stresses in U_2L_3 balance approximately the dead-load tensile stress, and the total stress in the member is zero.

Let the train advance upon the span. During this advancement the counter L_2U_3 comes into action, and when wheel 3 arrives at L_2 the live-load tensile stress reaches the maximum of 55.2, as shown in Article **112.** As the train advances from this position the tensile stress in the counter decreases, and when wheel 16 is 4 ft. on the span the member ceases to act. Thus, for a second time the total stress in L_2U_3 and in U_2L_3 is zero, as the following computations will prove.

The reaction at L_8 is

$$R_8 = \frac{13,073}{240} = 54.47$$

The floor-beam load at L_4 is

$$r_4 = \frac{70}{30} = 2.33$$

The floor-beam load at L_3 is

$$r_3 = \frac{1,785 + 230}{30} = 67.17$$

Moments about I of the forces on the right of the section through panel 2–3.

$$\begin{aligned} 54.47 \times 420 &= 22,877 \\ 2.33 \times 300 &= 700 \\ 67.17 \times 270 &= 18,135 \quad 18,835 \\ & \qquad 216)4,042 \\ & \qquad\qquad 18.7 = L.L. \\ 18.7 \times \frac{300}{397} &\quad 14.1 = I. \\ & \quad 32.8 = L.L. + I. \end{aligned}$$

The live-load and impact compressive stresses in U_2L_3 balance (approximately) the dead tensile stress and the resulting stress in U_2L_3 is zero. Thus when wheel 16 is 4 ft. on the span, the counter ceases to act and the main diagonal is about

to take stress as the train advances. This is evidently the greatest amount of train load which the truss supports when U_2L_3 is inactive, and is the correct position of the train for maximum live-load tensile stress in U_2L_2, which must balance the difference in the vertical components of the stresses in U_1U_2 and U_2U_3. Instead of solving by the method of joints we shall adopt the shorter method of passing a section through U_1U_2, U_2L_2 and L_2L_3, and balancing the moments of the forces on the right of the section about J, as follows:

$$\begin{aligned}
54.47 \times 300 &= \quad 16{,}341 \\
2.33 \times 180 &= \quad 420 \\
67.17 \times 150 &= 10{,}075 \quad 10{,}495 \\
&\qquad\qquad 120)\overline{5{,}846} \\
&\qquad\qquad\qquad 48.7 = L.L. \\
48.7 \times \frac{300}{397} &= \underline{36.8} = I. \\
&\qquad\qquad\quad 85.5 = L.L. + I.
\end{aligned}$$

Thus the maximum live-load and impact tensile stress in U_2L_2, when counters are used, is 85.5.

SEC. V. THE BALTIMORE TRUSS

116. The Baltimore Truss.—The Pratt truss shown in Fig. 120 has about the proper length of panel for economy, but the economic height of this truss gives too steep a slope for the diagonal web members. The slope of the diagonals in Fig. 121 is more nearly correct for an economic design but the panels are too long. The Baltimore truss (Fig. 122) is an evolution of the Pratt truss (Fig. 121); for if an intermediate floor beam is placed at the middle of each panel in Fig. 121 and supported by a *sub-vertical*, M_3L_3 for example, and the *sub-tie* M_3U_4 is added to prevent bending in U_2L_4, the Baltimore truss of Fig. 122 is the result. In the end panel the *sub-strut* M_1L_2 is used. Sub-struts instead of sub-ties are frequently used in the other panels, as shown in Fig. 92.

117. L_4L_6.—The center of moments for this member is at U_4 (Fig. 122) and the section is passed through the panel 4–5,

and it is clear that the influence line is the same as for L_4L_5 (Fig. 120) and L_4L_6 (Fig. 121). Hence, the criterion for maximum stress is the same for all three members. The critical wheel is at L_4, and one-third the total load on the span is placed on the segment L_0L_4. The maximum tensile stress occurring when wheel 13 is at L_4, is 374.7.

118. U_2U_4.—The stress in U_2U_4 (Fig. 121) is the same as that in L_4L_6, whereas in Fig. 122 the sub-tie M_3U_4 introduces a slight complication. The center of moments is at L_4, the

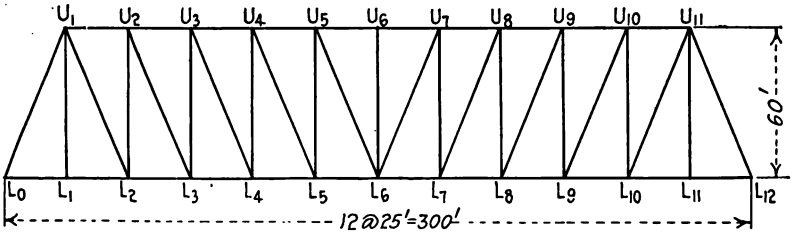


FIG. 120.

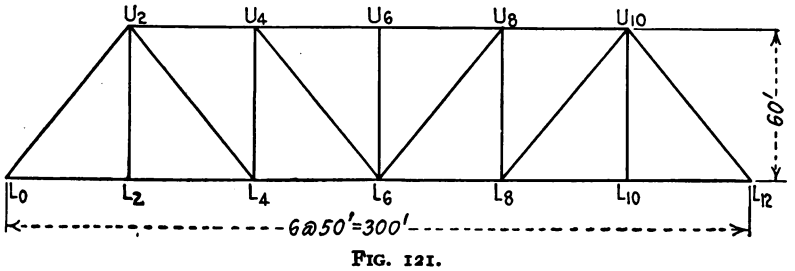


FIG. 121.

section is passed through the panel 2–3; and it is important to note that the floor beam at L_3 intervenes between the section and the center of moments.

As the load of one panel moves from L_{12} to L_3 (not L_4 as previously) the only force acting on the left of the section is the left reaction R_0, and the moment about L_4 is $100R_0$; hence QK = 100 ft.-lb., and OT is the influence line for moment about L_4 when the load of 1 lb. is between L_{12} and L_3. As the load of one panel moves from L_0 to L_2, the only force acting on the right of the section is the right reaction R_{12}; hence, OH

= 200 ft.-lb. and QS is the influence line for moment about L_4 when the load of 1 lb. is between L_0 and L_2.

$$TV = \frac{3}{4} \times 100 = 75$$

$$SU = \frac{1}{6} \times 200 = 33.3$$

Let the load of 1 lb. be in the panel 2–3 at a distance x from L_3, and let R_0 represent the reaction at L_0 and r_2 the floor beam load at L_2; then

$$R_0 = \frac{225 + x}{300} \text{ and } r_2 = \frac{x}{25}$$

Let M represent the algebraic sum of the moments about L_4 of all the forces on one side (either side) of the section, then

$$M = 100R_0 - 50r_2$$

$$M = \frac{225 + x}{3} - 2x = \frac{225 - 5x}{3}$$

This equation, being of the first degree, may be represented by a straight line. When the load of 1 lb. is at L_3, $x = 0$ and

$$M = \frac{225}{3} = 75 = TV$$

When the load of 1 lb. is at L_2, $x = 25$ and

$$M = \frac{100}{3} = 33.3 = SU$$

Since the influence line $OTSQ$ has three different slopes, we shall let

P_1 = the load on the segment L_0L_2.
P_2 = the load on the segment L_2L_3
P_3 = the load on the segment L_3L_{12}

The criterion for maximum moment about L_4 of all the forces acting on one side (either side) of the section through panel 2–3, developed in the same manner as explained in the preceding sections, is

$$2P_1 + 5P_2 \gtrless P_3$$

Adding $P_1 + P_2$ to both sides of the inequality, and letting $P_1 + P_2 + P_3 = P$, we have,

$$\frac{P}{3} \gtrless P_1 + 2P_2$$

The critical wheel is at L_3 and the position of the train for maximum stress in U_2U_4 should not differ materially from the position of maximum stress in L_4L_6, previously given.

Try wheel 9 at L_3.

The length of uniform load on the span is 164 ft.

$$\frac{P}{3} = 204$$

When wheel 9 is approaching L_3, wheel 5 is approaching L_2 and

$$P_1 = 70$$
$$2P_2 = 118$$

When wheel 9 has passed L_3, wheel 5 has passed L_2 and

$$P_1 = 90$$
$$2P_2 = 104$$

hence $$240 \gtrless \begin{matrix} 188 \\ 194 \end{matrix}$$

and wheel 9 at L_3 will not give a maximum stress in U_2U_4.

Try wheel 10 at L_3.

$$\frac{P}{3} = 209.3$$
$$P_1 = 90$$

When wheel 10 is approaching L_3, $P_2 = 52$; and when wheel 10 has passed L_3, $P_2 = 62$, hence

$$209.3 \gtrless \begin{matrix} 194 \\ 214 \end{matrix}$$

and the criterion is satisfied.

Wheel 11 at L_3 gives

$$215.3 \gtrless \begin{matrix} 188 \\ 228 \end{matrix}$$

Wheel 12 at L_3 gives

$$218.3 \gtrless \begin{matrix} 215 \\ 255 \end{matrix}$$

Wheel 13 at L_3 gives

$$221.3 \lessgtr \begin{matrix} 242 \\ 282 \end{matrix}$$

It is evident that wheels 10, 11 and 12 satisfy the criterion, and the algebraic sum of the moments about L_4 of all the forces acting on one side (either side) of the section through panel 2–3 must be computed for each condition to determine the maximum.

Wheel 10 *at* L_3.—The forces on the left of the section are the truss reaction R_0, the loads P_1 between L_0 and L_2, and the floor-beam load r_2 at L_2 resulting from the loads in panel 2–3.

$$R_0 = \frac{94{,}796}{300} = 315.99$$

The moment of P_1, about L_4 is

$$\begin{array}{r} 830 \\ 90 \times 58 = \underline{5{,}220} \\ 6{,}050 \end{array}$$

The floor-beam load at L_2 resulting from the loads in the panel is

$$r_2 = \frac{832}{25} = 33.28$$

$$\begin{array}{rl} 100R_0 = & 31{,}599 \\ \text{moment of } P_1 = & 6{,}050 \\ 50r_2 = & \underline{1{,}664 \quad 7{,}714} \\ & 60)\underline{23{,}885} \\ & 398.1 = \text{compressive stress in } U_2U_4 \end{array}$$

when wheel 10 is at L_3.

It is evident that these computations may be simplified as follows:

$$\begin{array}{rr} & 3)\underline{94{,}796} \\ & 31{,}599 \\ & 830 \\ 90 \times 58 = & 5{,}220 \\ 832 \times 2 = & \underline{1{,}664 \quad 7{,}714} \\ & 60)\underline{23{,}885} \\ & 398.1 \end{array}$$

Wheel 11 *at* L_3.

$$\begin{array}{rr} & 16{,}364 \\ 284 \times 180 = & 51{,}120 \\ 180^2 = & 32{,}400 \\ \hline & 3)\,99{,}884 \\ & 33{,}295 \\ \hline 2{,}155 & \\ 116 \times 52 = 6{,}032 & \\ 561 \times \ 2 = 1{,}122 & 9{,}309 \\ \hline & 60)\,23{,}986 \\ & 399.8 \end{array}$$

= compressive stress in U_2U_4 when wheel 11 is at L_3.

Wheel 12 *at* L_3.

$$\begin{array}{rr} & 16{,}364 \\ 284 \times 185 = & 52{,}540 \\ 185^2 = & 34{,}225 \\ \hline & 3)\,103{,}129 \\ & 34{,}376 \\ \hline 2{,}851 & \\ 129 \times 51 = 6{,}579 & \\ 503 \times \ 2 = 1{,}006 & 10{,}436 \\ \hline & 60)\,23{,}940 \\ & 399 \end{array}$$

= compressive stress in U_2U_4 when wheel 12 is at L_3.

Wheel 11 at L_3 evidently gives the largest maximum compressive stress in U_2U_4, although there is little difference in the stress for the three positions which give a maximum.

119. In Fig. 122 let $L_0L_2 = a$, $L_2L_3 = b$, $L_3L_4 = b$ and $L_4L_{12} = c$. Let P_1 = the load on the segment 0–2, P_2 = the load in panel 2–3, and P = the total load on the span. Show that the criterion for maximum stress in U_2U_4 is

$$\frac{P}{l} \gtrless \frac{P_1 + 2P_2}{l - c}$$

120. Suppose that the sub-tie M_3U_4 is omitted, and the sub-strut L_2M_3 is substituted. Let a, b and c represent the same

lengths as before. If P_1 = the load on the segment 0–3, P_2 = the load in panel 3–4, and P = the total load on the span, show that the criterion for maximum stress in L_2L_4 is

$$\frac{P}{l} \gtrless \frac{P_1 - P_2}{a}$$

121. U_2M_3.—The influence line is drawn for shear in panel 2–3, and is the same in all respects as the influence line for shear in the corresponding panel of Fig. 120. Hence the criterion is the same in both cases, or in other words one-twelfth of the total load on the span is placed in the panel for the maximum positive or negative shear. The stress in U_2M_3 (Fig. 122) will not be the same as the stress in U_2L_3 (Fig. 120), although the maximum shear in both cases is the same; for the two members do not have the same slope. The maximum positive shear (wheel 3 at L_3) is 222.9; the maximum negative shear (wheel 2 at L_2) is 13.2.

122. M_3L_4.—The influence line will be drawn for the vertical component of the stress. When the load of 1 lb. is at any point on the span except between L_2 and L_4, there is no stress in M_3L_3 or M_3U_4 and the stress in M_3L_4 is the same as the stress in U_2M_3. Consequently the influence line for M_3L_4 will be the same as for U_2M_3, except when the load of one pound is between L_2 and L_4.

$$TV = +\frac{2}{3}$$

$$US = -\frac{1}{6}$$

Let the load of 1 lb. be in panel 3–4 at the distance x from L_4. Let R_0 represent the truss reaction at L_0; r_3, the floor-beam load at L_3; and r_2 the floor-beam load at L_2; then

$$R_0 = \frac{200 + x}{300}, \; r_3 = \frac{x}{25} \text{ and } r_2 = 0$$

The web members meeting at M_3 are shown in Fig. 123. Let V represent the vertical component of the stress in M_3L_4, for which we are drawing the influence line. The vertical component of the stress in U_2M_3 is the shear in panel 2–3, which is R_0. Balance the moments of all the forces about U_2, thereby

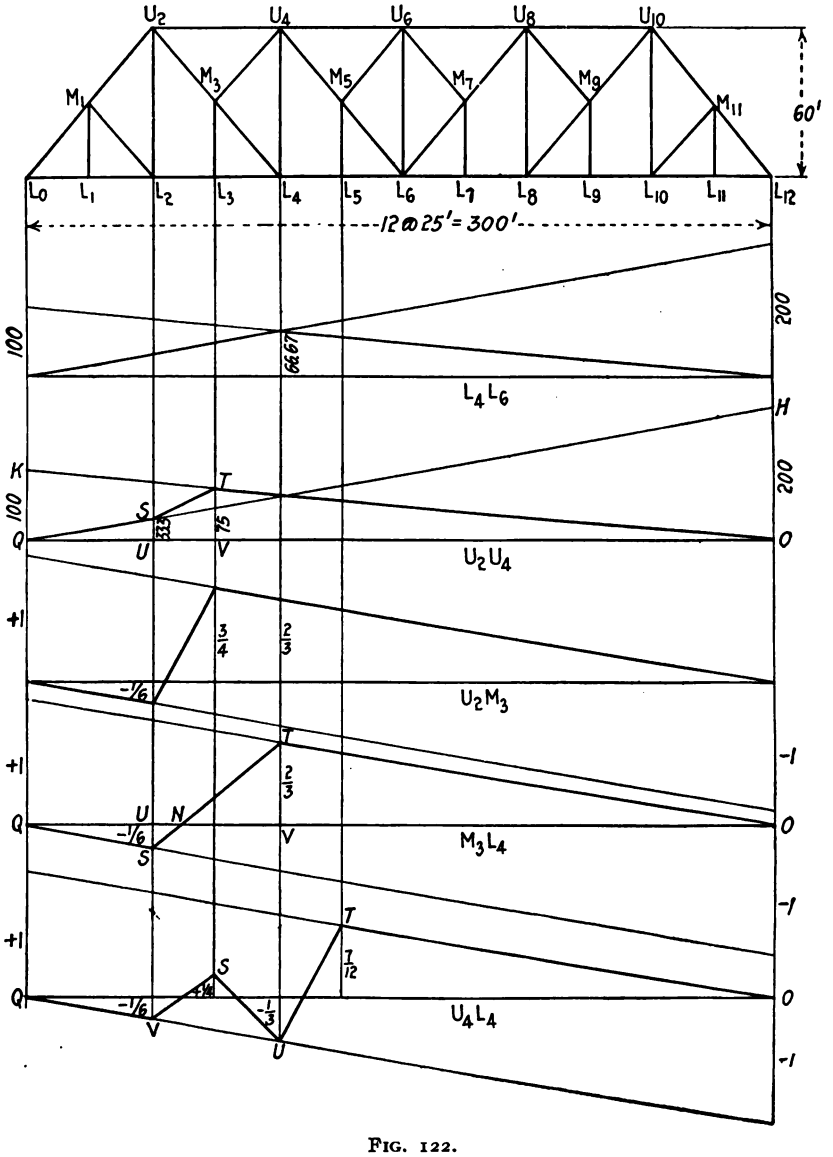

U2
U4
U6
U8
U10
M1
M3
M5
M7
M9
M11
60'
L0 L1 L2 L3 L4 L5 L6 L7 L8 L9 L10 L11 L12
12 @ 25' = 300'
100
200
6667
L4L6
H
K
T
S
533
75
Q
U
V
U2U4
+1
3/4
2/3
-1/6
U2M3
-1
N
M3L4
7/12
-1/3
U4L4
0

FIG. 122.

eliminating all the forces except the force at L_3 and the vertical component at U_4, whence the vertical component at U_4 is

$$\frac{25r_3}{50} = \frac{1}{2}r_3$$

Balance the vertical components

$$V + r_3 = R_0 + \frac{1}{2}r_3$$

$$V = R_0 - \frac{1}{2}r_3 = \frac{200 + x}{300} - \frac{x}{50}$$

$$V = \frac{200 - 5x}{300}$$

Now let the load of 1 lb. be in panel 2–3 at a distance x from L_4, then

$$R_0 = \frac{200 + x}{300}, r_3 = \frac{50 - x}{25}, \text{ and } r_2 = \frac{x - 25}{25}$$

The vertical component of the stress in U_2M_3 (Fig. 124) equals the shear in panel 2–3, which is $R_0 - r_2$, and

$$V + r_3 = R_0 - r_2 + \frac{1}{2}r_3$$

$$V = \frac{200 - 5x}{300}$$

Hence, when the load is at a distance x from L_4, whether in panel 3–4 or panel 2–3, the vertical component of the stress in M_3L_4 is given by the equation

$$V = \frac{200 - 5x}{300}$$

which may be represented by a straight line. When the load of 1 lb. is L_4, $x = 0$ and

$$V = +\frac{2}{3} = TV$$

When the load of 1 lb. is at L_2, $x = 50$ and

$$V = -\frac{1}{6} = US$$

Hence the straight line TS is the influence line for the vertical component of the stress in M_3L_4 when the load of 1 lb. is between L_2 and L_4.

It is clear that the influence line $OTNSQ$ is *also* the influence

line for shear in panel 2–4 (Fig. 121); or in other words, it is an influence line for the vertical component of the stress in U_2L_4.

Obviously then, the criterion for a maximum stress in M_3L_4 (Fig. 122) is the same as for maximum stress in U_2L_4 (Fig. 121), likewise the maximum stress is the same in both members. In order to show this more clearly, we shall compute the maximum tensile stress in U_2L_4 (Fig. 121).

Try wheel 5 at L_4.

23		16,364
200	$284 \times 114 =$	32,376
223	$114^2 =$	12,996
109		300)61,736
114		205.79 $= R_0$
2	$\frac{830}{50} =$	16.6 $= r_2$
228		189.19 = shear in panel 2–4
284		

6)512

$85.3 > 70$ O.K.
< 90

$$189.19 \times \frac{78.1}{50} = 295.5 = \text{maximum tensile stress in } U_2L_4.$$

If a section is passed through panel 3–4 (Fig. 122), it is evident that the vertical forces acting on the left of the section must balance. The vertical component of M_3U_4 acting upward on the left portion of the truss is $\frac{1}{2}r_3$. The forces acting upward on the left of the section are R_0 and $\frac{1}{2}r_3$, and they are balanced by the floor-beam loads on the left of the section and the vertical component V in M_3L_4; hence

$$R_0 + \frac{1}{2}r_3 = r_2 + r_3 + V$$

When wheel 5 is at L_4, wheel 1 is in panel 3–4

$$r_2 = 0$$
$$r_3 = 33.2$$
$$R_0 = 205.79$$

hence $$V = R_0 - \frac{1}{2}r_3 = 189.19$$

Thus it is clear that the tensile stress in M_3L_4 (Fig. 122) is the same as in U_2L_4 (Fig. 121).

124. U_4L_4.—The influence line for stress in U_4L_4, when the sub-members M_3L_3 and M_3U_4 are omitted, is $OTUQ$. The presence of the sub-members alters the influence line between L_2 and L_4. When the load of 1 lb. is between L_2 and L_4 the sub-members take stress, and the stress in U_4L_4 when the load of 1 lb. is at L_3 is $\frac{1}{4}$ lb. compressive. It may be shown by the method used on previous occasions that the influence

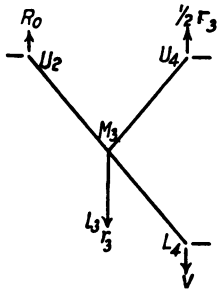


FIG. 123.

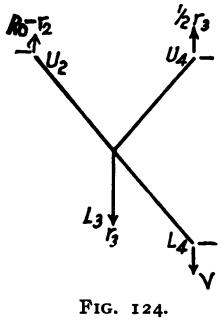


FIG. 124.

line between panel points is a straight line, therefore $OTUSVQ$ is the influence line for stress in U_4L_4.

Let P_1 = the load on L_0L_2
P_2 = the load on L_2L_3
P_3 = the load on L_3L_4
P_4 = the load on L_4L_5
P_5 = the load on L_5L_{12}
P = total load on the span

The criterions, when developed as in preceding cases, will show that the compressive stress in U_4L_4 will be increasing as the loads move to the left as long as

	$P_1 + 7P_3 + P_5$	$> 5P_2 + 11P_4$
add	$P_2 + P_4$	$= P_2 + P_4$
or as long as	$P + 6P_3$	$> 6P_2 + 12P_4$
or as long as	$\frac{P}{6} + P_3 >$	$P_2 + 2P_4$

Two conditions of loading should be investigated. The train

may advance from the right until a few loads pass L_5, in which case P_1, P_2 and P_3 are zero, and the criterion becomes

$$\frac{P}{12} \gtrless P_4$$

which is obviously the same as the criterion for U_4M_5. Or the train may advance until the compressive area of the influence line at S is covered. Since the compressive area at S is smaller than the tensile area at U, it is doubtful if the second position of the train will give a larger compressive stress than the first; although if large wheels are near L_3 and smaller wheels near L_4, the difference in areas will not be so significant. For the first position of this train, wheel 3 at L_5 satisfies the criterion

$$\frac{P}{12} \gtrless P_4$$

and the maximum compressive stress in U_4L_4 is 140.94.

For the second position of this train the criterion is

$$\frac{P}{6} + P_3 \gtrless P_2 + 2P_4$$

and the critical wheel will be at either L_3 or L_5.

Try wheel 3 at L_3

WHEEL 3 APPROACHING L_3

$P_1 = 0$
$P_2 = 30$
$P_3 = 86$
$P_4 = 36$
$P_5 = 390$
$P = 542$

$\frac{P}{6} = 90.3 \qquad P_2 = 30$

$P_3 = 86 \qquad 2P_4 = 72$

$176.3 > \qquad 102$

WHEEL 3 PASSING L_3

$P_1 = 0$
$P_2 = 50$
$P_3 = 66$
$P_4 = 36$
$P_5 = 390$
$P = 542$

$\frac{P}{6} = 90.3 \qquad P_2 = 50$

$P_3 = 66 \qquad 2P_4 = 72$

$156.3 > \qquad 122$

In either case the compressive stress in U_4L_4 is increasing.

Try wheel 4 at L_3

WHEEL 4 APPROACHING L_3

$\frac{P}{6} = 92 \qquad P_2 = 50$

$P_3 = 66 \qquad 2P_4 = 112$

$158 \qquad < \quad 162$

WHEEL 4 PASSING L_3

$\frac{P}{6} = 92 \qquad P_2 = 70$

$P_3 = 59 \qquad 2P_4 = 86$

$151 \qquad < \quad 156$

In either case the compressive stress in U_4L_4 is decreasing. Since the stress is increasing when wheel 3 passes L_3, and is decreasing when wheel 4 arrives at L_3; there must be an intermediate position for which the stress is a maximum. Try wheel 11 at L_5

WHEEL 11 APPROACHING L_5	WHEEL 11 PASSING L_5
$\frac{P}{6} = 90.7 \quad P_2 = 50$	$P_6 = 90.7 \quad P_2 = 50$
$P_3 = \frac{66}{156.7} > \frac{2P_4 = 72}{122}$	$P_3 = \frac{66}{156.7} < \frac{2P_4 = 112}{162}$

Wheel 11 at L_5 satisfies the criterion for maximum compressive stress. Pass a section cutting the members U_2U_4, M_3U_4, U_4L_4 and L_4L_5 and consider the forces acting on the left of the section. The forces acting upward are the truss reaction $R_0 = 233.95$, and the vertical component of the stress in $M_3U_4 = \frac{1}{2}r_3 = 36.54$. The forces acting downward are the sum of the wheel loads on the left of L_4, which is 116, and the floor-beam load at L_4 due to the loads in panel 4-5, which is 22.44. Hence the compressive stress in U_4L_4 is 132.05. This stress is somewhat less than that previously determined for wheel 3 at L_5.

For the maximum tensile stress in U_4L_4, the train approaches from the left and it is obvious from the influence line that the train will advance until the large wheels of the front engine are at L_4.

Try wheel 3 at L_4

WHEEL 3 APPROACHING L_4	WHEEL 3 PASSING L_4
$P_1 = 140$	$P_1 = 140$
$P_2 = 36$	$P_2 = 36$
$P_3 = 86$	$P_3 = 66$
$P_4 = 30$	$P_4 = 50$
$P_5 = 0$	$P_5 = 0$
$P = 292$	$P = 292$
$\frac{P}{6} = 48.7 \quad P_2 = 36$	$\frac{P}{6} = 48.7 \quad P_2 = 36$
$P_3 = \frac{86}{134.7} > \frac{2P_4 = 60}{96}$	$P_3 = \frac{66}{124.7} < \frac{2P_4 = 100}{136}$

Wheel 3 at L_4 satisfies the criterion for maximum tensile stress in U_4L_4. Consider the forces acting on the right of the section through U_2U_4, M_3U_4, U_4L_4 and L_4L_5. The forces

acting upward are the truss reaction $R_{12} = 58.39$ and the floor beam load $r_5 = 9.2$. The vertical component of $M_3U_4 = \frac{1}{2}r_3 = 27.68$ acts downward. Hence the tensile stress in U_4L_4 when wheel 3 is at L_4, is 21.51.

It may be interesting to investigate the maximum tensile stress in U_4L_4 when the train, approaching from the left, has advanced until only a few loads are on panel 2–3; in which case P_3, P_4 and P_5 are zero and the criterion reduces to

$$\frac{P}{6} \leqslant P_2$$

Wheel 2 at L_2 satisfies this criterion. $R_{12} = 16.45$, $r_3 = 3.2$ and the maximum tensile stress in U_4L for this position is $16.45 - 1.6 = 14.85$; which is less than in the preceding case, when wheel 3 has advanced to L_4.

125. U_2L_2.—When the 1 lb. load is at L_1, the vertical component of the stress in M_1L_2 causes a tensile stress of $\frac{1}{2}$ lb. in U_2L_2. Hence the influence line is a triangle with the apex at L_2 and a base line from L_0 to L_3. For maximum tensile stress the train comes on the span at L_0 and one-third of the total load is in panel 2–3. The stress equals the algebraic sum of the moments about L_0 of the truss reaction at L_{12} and the floor-beam load at L_3, divided by the distance L_0L_2.

In case the main diagonals are designed to carry a tensile stress only, and it is found that counters are required (for example in panel 4–6) the member L_4M_5 is added. The members L_4M_5 and M_5U_6 then become the main diagonals. The diagonal U_4M_5 becomes a sub-tie and the vertical component of its stress is one-half the stress in M_5L_5, and the member M_5L_6 is not acting at all.

SEC. VI. THE PENNSYLVANIA TRUSS

126. Pennsylvania Truss.—The influence lines for several members of a Pennsylvania truss are shown in Fig. 125. This truss is an evolution of the Baltimore truss, and bears the same relation to it that exists between the Parker and the Pratt trusses.

127. U_2U_4.—The influence line, the criterion and position of a train for maximum stress in this member are the same as for the corresponding member of the Baltimore truss in Fig.

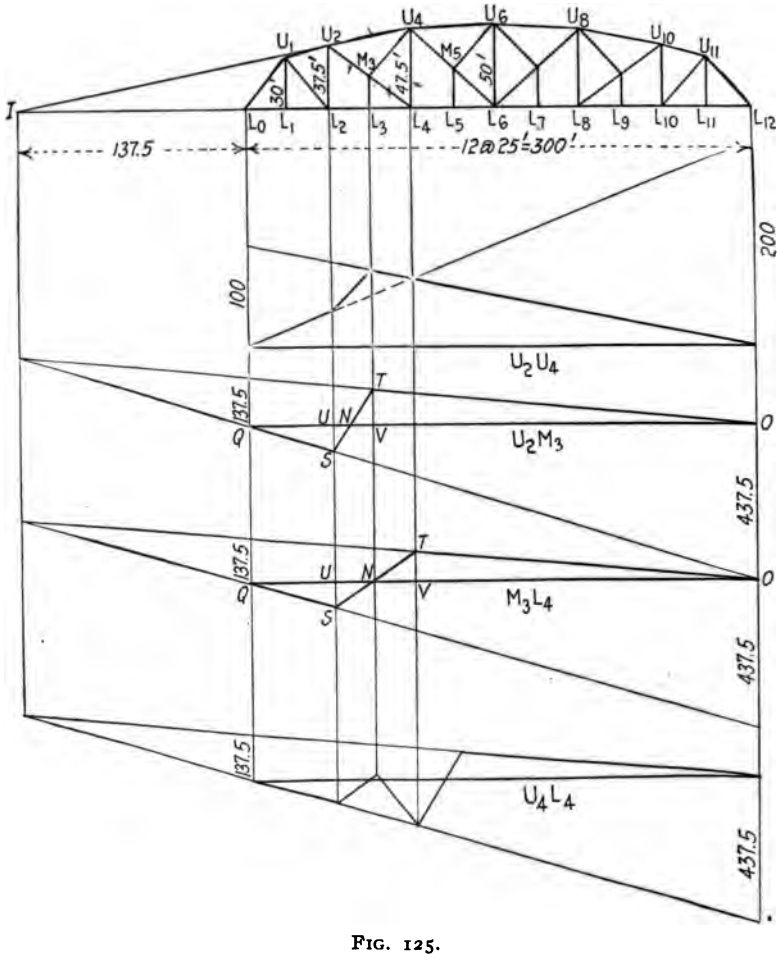


FIG. 125.

122. The stresses differ only because the lever arms from L_4 to the members are different in the two trusses.

128. U_2M_3.—The section is passed through panel 2–3, and the chord members intersect at I. In the Baltimore truss, one-twelfth of the total load on the span is placed in panel 2–3 for maximum tensile or compressive stress. In the Pennsly-

vania truss, the train covers the right portion of the truss, with the critical wheel at L_3 for a maximum tensile stress, and the moment about I of all the forces on the left of the section is

$$M_I = 137.5R_0 - 187.5r_2$$

Let P_2 represent the load in panel 2–3, and P the total load on the span; then the criterion, developed as in Article **111** or **114**, is

$$P_2 \lessgtr \frac{1}{12}\frac{137.5}{187.5}P$$

or

$$P_2 \lessgtr 0.061P$$

The criterion may also be determined by the ratio,

$$\frac{NV}{NO} = \frac{14.6}{239.6} = 0.061$$

For a maximum compressive stress the left portion of the truss is loaded with the critical wheel at L_2, and

$$M_I = 437.5R_{12} - 212.5r_3$$

and the criterion is $P_2 \lessgtr \frac{1}{12}\frac{437.5}{212.5}$

or

$$P_2 \lessgtr 0.172P$$

This criterion may also be determined by the ratio,

$$\frac{NU}{NQ} = \frac{10.4}{60.4} = 0.172$$

129. M_3L_4.—As in the Baltimore truss, the stress in this member is the same as the stress in U_2L_4 when the sub-members M_3L_3 and M_3U_4 are removed. This is clearly shown by the influence line. With the sub-members omitted, the maximum tensile stress in U_2L_4 occurs when the right portion of the truss is loaded and the critical wheel is at L_4, and

$$M_I = 137.5R_0 - 187.5r_2$$

Let P_2 represent the load on L_2L_4, and P the total load; then the criterion is

$$P_2 = \frac{1}{6}\frac{137.5}{187.5}P$$

$$\text{or } P_2 = 0.122P$$

Likewise

$$\frac{NV}{NO} = \frac{27.8}{227.8} = 0.122$$

For a maximum compressive stress in U_2L_4, the left portion of the truss is loaded with the critical wheel at L_2, and

$$M_1 = 437.5R_{12} - 237.5r_4$$

Hence the criterion is $P_2 \gtrless \frac{1}{6}\frac{437.5}{237.5}P$

or $P_2 \gtrless 0.307P$

Likewise $$\frac{NU}{NQ} = \frac{22.2}{72.2} = 0.307$$

130. U_4L_4.—For maximum compression, the right portion of the truss is loaded with the critical wheel at L_5 and the criterion is

$$P_2 \gtrless \frac{1}{12}\frac{137.5}{237.5}P$$

or $P_2 \gtrless 0.048P$

Likewise $$\frac{NV}{NO} = \frac{8.87}{183.87} = 0.048$$

The criterion for maximum tensile stress requires a special treatment similar to that given for the corresponding case in the Baltimore truss. Since the compressive area of the influence line at L_3 is comparatively small, it is evident that the train in approaching from the left should extend into panel 4–5, with wheel 2 or 3 at L_4.

Counters are treated in a similar manner as outlined for the Baltimore truss.

SEC. VII. EQUIVALENT UNIFORM LOADS

131. In the previous sections of this chapter the shear and moment at a given section in a beam or girder, also the stress in a given member of a truss caused by a train load, were computed by means of a moment table Fig. (100). If a *uniform* live load were used, the computations would be greatly simplified; as will be seen from the following examples.

132. Examples.—Consider a uniform live load of 3,000 lb. per linear foot in connection with the beam of Fig. 97. The influence line shows that the uniform live load will cover the segment BC for a maximum positive shear at C; and the segment AC for a maximum negative shear at C. Let AC = 10 ft., and

$BC = 15$ ft.; then $TN = +0.6$ and $NS = -0.4$. When the live load extends from B to C, the shear at C equals the reaction at A; or

$$F_C = R_A = \frac{3{,}000 \times 15 \times 15}{2 \times 25} = 13{,}500 \text{ lb.}$$

The shear may also be obtained by taking the product of the area OTN and the intensity of the uniform load per foot, thus

$$F_C = \frac{0.6 \times 15}{2} \times 3{,}000 = 13{,}500 \text{ lb.}$$

Similarly the maximum negative shear at C, occurring when the live load extends from A to C, is

$$F_C = \frac{-0.4 \times 10}{2} \times 3{,}000 = -6{,}000 \text{ lb.}$$

In Fig. 98 the influence line shows that a uniform live-load covers the whole span for a maximum bending moment at C. For a uniform live load of 2,000 lb. per linear foot the bending moment at C is

$$M_C = \frac{2{,}000}{2} \times 10 \times 15 = 150{,}000 \text{ ft.-lb.}$$

The bending moment may also be determined by taking the product of the area QNO and the intensity of the uniform load per foot, thus

$$M_C = \frac{6 \times 25}{2} \times 2{,}000 = 150{,}000 \text{ ft.-lb.}$$

In Fig. 108 the influence line shows that a uniform live load extends from O to N for a maximum positive shear in panel 1–2. $NV = 16.67$ ft. Let the intensity of the uniform live load be 3,600 lb. per linear foot, then,

$$R_0 = \frac{3{,}600 \times 66.67^2}{2 \times 100} = 80{,}000 \text{ lb.}$$

$$r_1 = \frac{3{,}600 \times 16.67^2}{2 \times 25} = 20{,}000 \text{ lb.}$$

and the shear in panel 1–2 is

$$F = R_0 - r_1 = 60{,}000 \text{ lb.}$$

The shear in the panel may also be found by taking the

product of the area NTO and the intensity of the uniform load per foot, thus

$$F = \frac{66.67 \times 0.5}{2} \times 3{,}600 = 60{,}000 \text{ lb.}$$

133. General Considerations.—In Article **103** the maximum positive shear in panel 1-2 of the truss in Fig. 109 was found to be 47,760 lb. for an E-40 train. Let q represent the intensity of the equivalent uniform live load which will cause the same shear, then

$$q \times \text{area } NTO = 47{,}760$$

$$\text{area } NTO = \frac{66.67 \times 0.5}{2} = 16.67$$

$$q = \frac{47{,}760}{16.67} = 2{,}866 \text{ lb. per linear foot}$$

Hence it is clear that a uniform live load of 2,866 lb. per linear foot for one truss will cause the same maximum positive shear in panel 1–2 of the truss in Fig. 109, as an E-40 train.

The maximum negative shear in panel 1–2 for an E-40 train was found to be 14,230 lb. Let q represent the intensity of the equivalent uniform live load which will cause the same negative shear, then

$$q \times \text{area } QSN = -14{,}230$$

$$\text{area } QSN = -4.167$$

$$q = 3415 \text{ lb. per linear foot}$$

In Article **98** the maximum moment for the stress in U_1U_2 (Fig. 106) for an E-40 train was found to be 3,219,000 ft.-lb. The area of the influence line (shown in Fig. 105) is 1,250; hence the intensity of the equivalent uniform load for this member is

$$q = \frac{3{,}219{,}000}{1{,}250} = 2{,}575 \text{ lb. per linear foot}$$

In Article **112** the maximum value for M_l for the member U_2L_3 (Fig. 117) was found to be 24,122,000 ft.-lb. The area of the influence line NTO is 9,309; hence the intensity of the equivalent uniform load for maximum tension in this member is

$$q = \frac{24{,}122{,}000}{9{,}309} = 2{,}591 \text{ lb. per linear foot.}$$

From a consideration of the examples just given, it is apparent that the stresses in a truss might be very quickly computed if a uniform load could be substituted for the E-40 train load. The only hindrance to this substitution lies in the fact that the equivalent uniform load is not the same for all members of a

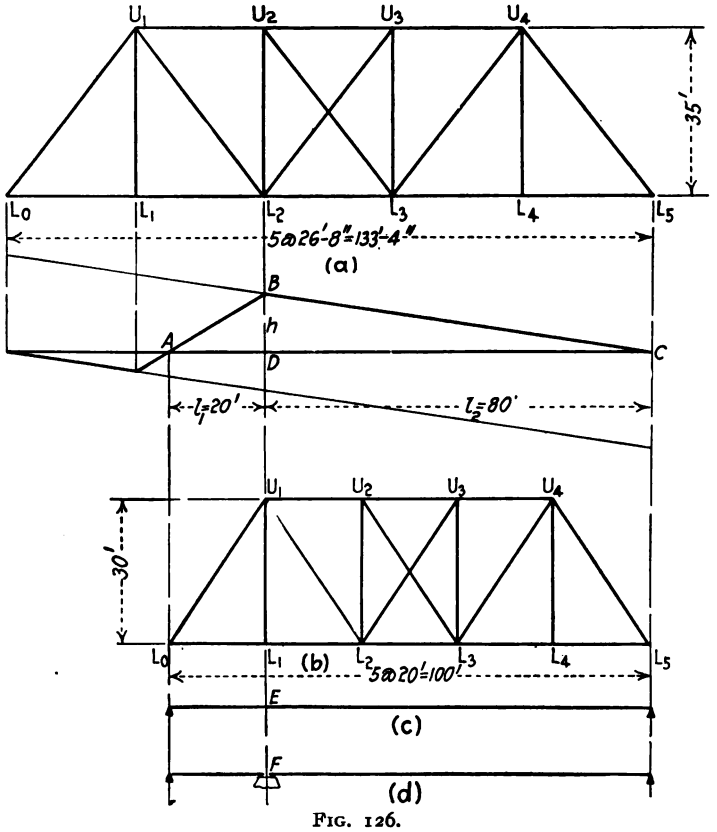


FIG. 126.

truss. In fact, the equivalent uniform load for the tensile stress in a web member is not the same as for the compressive stress in the same member, as was clearly shown.

134. Triangular Influence Diagrams.—Nearly all the influence line diagrams which have been considered in this chapter are triangular as shown in Fig. 126. It is clear that the triangle ABC, in which $l_1 = 20$ ft., and $l_2 = 80$ ft., may represent

the influence line for determining the following quantities for an E-40 train by varying the altitude h.

Let s = maximum positive shear in panel 1–2, truss (a)
t = maximum tension in U_1L_2 truss (a)
u = maximum stress in L_0U_1 truss (b)
v = maximum shear in panel 0–1 truss (b)
w = maximum stress in L_0L_2 truss (b)
x = maximum bending moment at E, beam (c)
y = maximum pier reaction at F, span (d)

$$\text{then } h = \begin{cases} 0.6 \text{ lb. for } s \\ 0.76 \text{ lb. for } t \\ 0.8 \text{ lb. for } u \\ 0.96 \text{ lb. for } v \\ 0.53 \text{ lb. for } w \\ 16 \text{ ft.-lb. for } x \\ 1.0 \text{ lb. for } y \end{cases}$$

Since l_1 and l_2 are constants in all these cases, the criterion for a maximum for all quantities (s to y) is the same. This criterion places wheel 4 on the vertical through D, and by the use of the moment table, the value of each quantity may be found as given below:

s = 81,982.5 lb.
t = 103,844 lb.
u = 109,310 lb.
v = 131,172 lb.
w = 72,417 lb.
x = 2,186,200 ft.-lb
y = 136,637.5 lb.

Instead of using the moment table these quantities could have been determined by taking the sum of the products of each wheel load and its corresponding ordinate in the influence line diagram, when the proper value for h is taken. Hence it is seen that the sum of the products in each case is proportional to the corresponding value of h, as an inspection will prove; thus for example, in the case of s and u,

$$81{,}982.5 : 0.6 :: 109{,}310 : 0.8$$

Since l_1 and l_2 are constants, it is clear that the area of the influence line diagram ABC for each quantity is *also* proportional to the corresponding value of h; and from this it follows that each quantity (s to y) is proportional to the area of its influence line diagram; or in other words, the ratio between each quantity and its influence line area is constant. This constant ratio is the intensity of the equivalent uniform load which corresponds to any triangular influence line diagram for which $l_1 = 20$ ft., and $l_2 = 80$ ft. Let q represent the intensity of the equivalent uniform load corresponding to an E-40 train load. From the previous discussion it is clear that q is a function of l_1 and l_2 and independent of h.

135. Table of Equivalent Uniform Loads.—Table I[1] gives the values of q, the equivalent uniform load per linear foot per rail for an E-40 loading, for various lengths of l_1 and l_2 in multiples of 5 ft. The values of q will be found correct in most instances to the third significant figure, and are thus accurate to within 1 per cent. The corresponding values of q for any other Cooper's standard loading are directly proportional. The following example will be given to illustrate the use of the table. In Article **112** the maximum tensile stress in U_2L_3 (Fig. 117) was found to be 111.7. The influence line NTO is drawn for M_1; $NV = 15.5 = l_1$; $VO = 150 = l_2$; and the area is 9,309. From Table I, $q = 2.590$ expressed in 1,000 lb.-units.

$$M_1 = 9{,}309 \times 2.59 = 24{,}110$$

$$U_2L_3 = \frac{24{,}110}{216} = 111.6$$

This gives a reasonably close check.

For the maximum compressive stress the area NQS is 3,911; $NU = 14.5 = l_1$; $UQ = 60 = l_2$; hence $q = 2.820$ and the maximum compressive stress is

$$\frac{3{,}911 \times 2.82}{216} = 51$$

[1] This table is taken from p. 112, 1st ed., of "Live-load Stresses in Railway Bridges," by GEORGE E. BEGGS, published by John Wiley & Sons. Professor Beggs has kindly granted permission to reproduce this table. There are many other tables in this book which will be found very useful for ready reference to the designing engineer.

TABLE I.—EQUIVALENT UNIFORM LOADS FOR COOPER'S E-40 LOADING VALUES IN POUNDS PER LINEAR FOOT PER RAIL

Longer segment l_2	Shorter segment l_1											
	0	5	10	15	20	25	30	35	40	45	50	55
250	2,500	2,450	2,430	2,410	2,380	2,370	2,350	2,330	2,310	2,300	2,290	2,270
225	2,550	2,500	2,460	2,450	2,430	2,400	2,380	2,360	2,340	2,320	2,310	2,300
200	2,610	2,540	2,500	2,490	2,460	2,440	2,420	2,390	2,370	2,350	2,340	2,320
175	2,680	2,610	2,550	2,540	2,510	2,490	2,460	2,420	2,400	2,380	2,360	2,340
160	2,730	2,630	2,590	2,570	2,540	2,510	2,480	2,450	2,420	2,400	2,380	2,370
150	2,760	2,670	2,620	2,590	2,570	2,540	2,500	2,460	2,430	2,420	2,400	2,380
140	2,800	2,700	2,650	2,620	2,580	2,560	2,520	2,490	2,450	2,430	2,420	2,400
130	2,850	2,740	2,670	2,650	2,610	2,580	2,540	2,510	2,470	2,450	2,430	2,420
120	2,900	2,770	2,710	2,680	2,640	2,610	2,560	2,530	2,490	2,460	2,450	2,430
110	2,940	2,810	2,740	2,710	2,660	2,630	2,580	2,550	2,500	2,490	2,460	2,460
100	3,000	2,850	2,780	2,740	2,690	2,660	2,610	2,570	2,530	2,510	2,500	2,480
95	3,020	2,880	2,800	2,760	2,700	2,670	2,620	2,580	2,560	2,540	2,520	2,500
90	3,050	2,890	2,810	2,770	2,720	2,680	2,630	2,620	2,590	2,570	2,550	2,540
85	3,080	2,920	2,820	2,780	2,730	2,700	2,640	2,640	2,620	2,580	2,570	2,550
80	3,110	2,920	2,840	2,790	2,740	2,710	2,670	2,660	2,620	2,610	2,580	2,570
75	3,140	2,940	2,860	2,800	2,740	2,700	2,670	2,660	2,640	2,620	2,600	2,580
70	3,160	2,940	2,870	2,810	2,750	2,700	2,670	2,660	2,650	2,620	2,600	2,580
65	3,190	2,960	2,870	2,810	2,760	2,700	2,670	2,660	2,650	2,620	2,600	2,580
60	3,270	3,020	2,880	2,820	2,750	2,700	2,660	2,640	2,630	2,610	2,590	2,580
55	3,370	3,090	2,930	2,840	2,760	2,700	2,660	2,650	2,620	2,600	2,560	2,550
50	3,490	3,180	3,000	2,910	2,800	2,740	2,700	2,670	2,630	2,600	2,580	
45	3,630	3,260	3,080	2,980	2,870	2,780	2,740	2,710	2,670	2,640		
40	3,770	3,350	3,180	3,060	2,930	2,840	2,780	2,740	2,700			
35	3,960	3,450	3,260	3,120	3,010	2,900	2,840	2,790				
30	4,200	3,610	3,380	3,200	3,060	2,960	2,880					
25	4,540	3,770	3,520	3,320	3,150	3,020						
20	5,000	4,000	3,730	3,450	3,280							
15	5,336	4,000	4,000	3,650								
10	6,000	4,000	4,000									
5	8,000	4,000										

TABLE I.—EQUIVALENT UNIFORM LOADS FOR COOPER'S E-40 LOADING VALUES IN POUNDS PER LINEAR FOOT PER RAIL

Longer segment l_2	Shorter segment l_1												
	60	65	70	75	80	85	90	95	100	110	120	130	140
250	2,260	2,260	2,250	2,250	2,240	2,230	2,220	2,220	2,210	2,200	2,180	2,160	2,140
225	2,290	2,280	2,270	2,270	2,260	2,260	2,250	2,240	2,220	2,220	2,180	2,170	2,150
200	2,310	2,300	2,290	2,290	2,280	2,280	2,270	2,260	2,250	2,230	2,200	2,180	2,160
175	2,340	2,320	2,320	2,320	2,310	2,300	2,290	2,280	2,270	2,240	2,210	2,200	2,180
160	2,350	2,340	2,340	2,340	2,330	2,320	2,310	2,300	2,280	2,260	2,230	2,210	2,180
150	2,370	2,350	2,360	2,350	2,340	2,340	2,330	2,300	2,300	2,270	2,240	2,220	2,190
140	2,380	2,380	2,370	2,360	2,360	2,350	2,340	2,320	2,310	2,280	2,250	2,220	2,200
130	2,400	2,390	2,390	2,380	2,380	2,370	2,350	2,340	2,330	2,290	2,260	2,230	
120	2,420	2,410	2,410	2,400	2,400	2,370	2,370	2,350	2,340	2,300	2,270		
110	2,440	2,420	2,420	2,420	2,420	2,400	2,390	2,380	2,350	2,320			
100	2,460	2,460	2,450	2,440	2,440	2,420	2,410	2,390	2,380				
95	2,500	2,480	2,460	2,460	2,450	2,440	2,420	2,400					
90	2,510	2,500	2,480	2,460	2,460	2,450	2,430						
85	2,530	2,510	2,500	2,490	2,470	2,460							
80	2,550	2,540	2,520	2,500	2,490								
75	2,560	2,540	2,530	2,510									
70	2,560	2,540	2,530										
65	2,560	2,540											
60	2,550												

CHAPTER V

DEFLECTION OF BEAMS

SEC. I. THE AREA-MOMENT METHOD

136. General Considerations.—Whenever vertical loads are applied between the supports of a horizontal beam, the top fibers are shortened and the bottom fibers are lengthened by the bending stresses. These changes in length, or deformations, bend the beam which was originally straight into a curved shape; the top surface becoming concave, and the bottom convex. The neutral axis of the beam in its bent condition is called the elastic curve. The vertical distance through which a point on the elastic curve has been moved by this bending is called the deflection. The distortion of the fibers by the shearing stresses also contributes to the deflection, but the additional amount is generally too small to have any practical value. The structural engineer frequently desires to determine the deflection of a beam or girder at one or more points in its length. This in itself makes a study of deflections desirable; but a more important use for the theory involved is its application to statically indeterminate structures.

There are several methods by which deflections caused by bending may be determined. In the oldest and most widely known method, the second differential equation of the elastic curve is derived. This equation must be integrated twice before the deflection at any point may be found. The method is long and greatly involved, except for the simplest conditions of loading. The simpler and less known method of area-moments establishes a relation between a tangent to the elastic curve and the bending moment diagram.

137. Method of Area-moments.—Let A and B (Fig. 127) represent any two points on the neutral axis of a beam, which is bent by any arrangement of loading. Through A and B draw the tangents AD and BC intersecting at C, and the

normals AI and BI intersecting at I. Then $\angle AIB = \angle BCD = \phi$. Let $QPRS$ represent the bending moment diagram for the portion of the beam between A and B. Let $EFHG$ represent an element of the beam between two right sections EG and

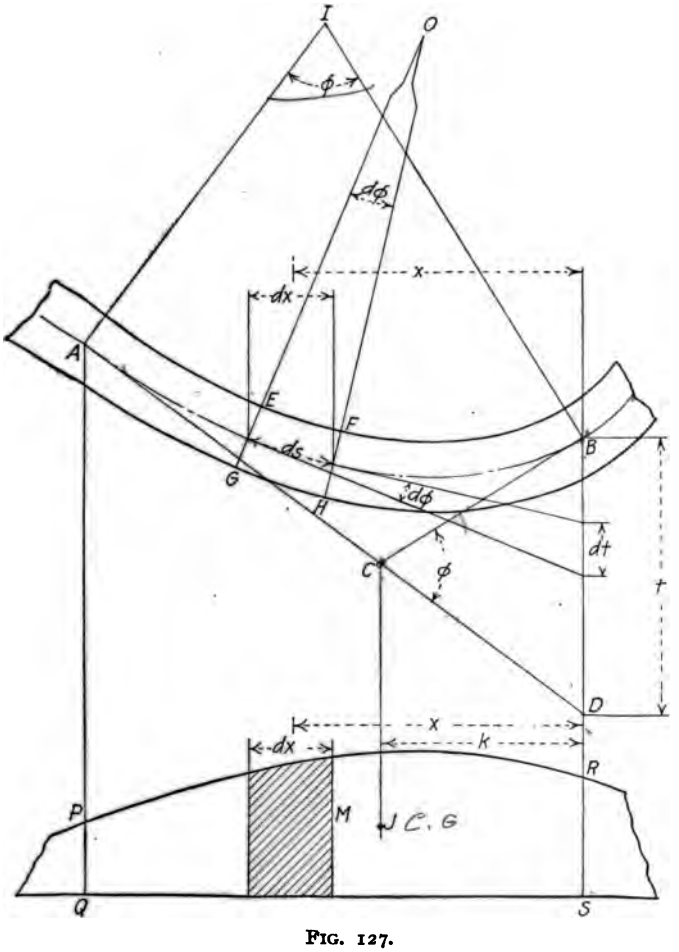


FIG. 127.

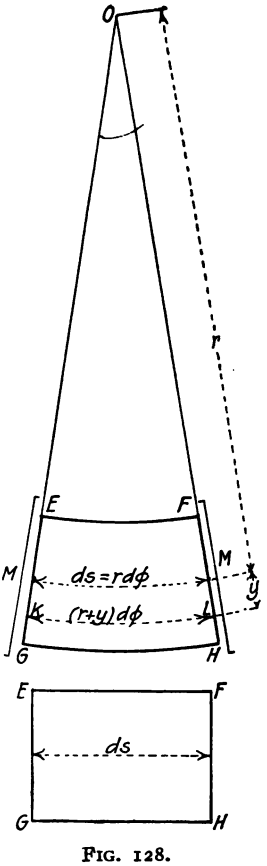


FIG. 128.

FH (drawn to a larger scale in Fig. 128) which were parallel and a distance ds apart before the element was bent by the bending moment M. Let r (Fig. 128) represent the radius of curvature of the neutral axis for this element. The fiber at the neutral axis remains unchanged in length, while the fiber KL at a

distance y below the neutral plane has been increased in length from $ds = rd\phi$ to $(r + y)d\phi$. Hence the total strain (change in length) in the length ds is $yd\phi$ and the unit strain is $\frac{yd\phi}{ds}$. Let f represent the unit stress on the fiber KL; and let E represent the modulus of elasticity.

Then $$E = \frac{f}{\frac{yd\phi}{ds}} \text{ or } f = \frac{Eyd\phi}{ds}$$

Let I = the moment of inertia of the cross-section about the neutral plane; then

$$f = \frac{My}{I}$$

whence $$\frac{Eyd\phi}{ds} = \frac{My}{I}$$

or $$d\phi = \frac{Mds}{EI}$$

and $$\phi = \int_A^B d\phi = \int_A^B \frac{Mds}{EI}$$

If the beam in its natural state is straight (not arched) and is properly designed, the curvature will be so slight that ds may be replaced by dx, allowing the integration to be made horizontally between A and B instead of along the path of the elastic curve. Then

$$\phi = \int_A^B \frac{Mdx}{EI}$$

If the beam is homogeneous and has a uniform cross-section, E and I are constants, and the equation may be written thus:

$$\phi = \frac{1}{EI}\int_A^B Mdx \,. \qquad (1)$$

The expression Mdx represents the area of the cross-hatched element in bending moment diagram. Hence the integral expression $\int_A^B Mdx$ is the area of the M-diagram between the ordinates RS and PQ, and if this area is divided by EI, the quotient is the angle ϕ. If M is expressed in inch-pounds, the area Mdx is expressed in inch^2-pounds. If E is expressed in pound/inches^2, and I in inches^4; then EI is also expressed in

inch^2-pounds, and the angle ϕ is a ratio. In any practical beam ϕ is comparatively very small; hence, when the tangent CB (Fig. 127) is horizontal, the ratio ϕ may be taken as the slope of the tangent AD. Likewise, when AD is horizontal, the ratio ϕ may be taken as the slope of the tangent CB. From this analysis the first principle may be deduced.

138. First Principle.—*If tangents are drawn through any two points on the elastic curve of a homogeneous beam of uniform cross-section, the angle which one tangent makes with the other tangent equals the area of the* M*-diagram between the two points, divided by* EI.

Now imagine that the unstrained position of the beam was in the direction AD, and that the beam was subsequently bent so that the point D moved to B; the point A remaining stationary. This movement is caused by the bending of all the elements from A to B. The bending of the element $EFGH$ causes the point, in its travel from D to B, to move a distance $dt = x d\phi$. Since the curvature is comparatively small, the path of the point moving from D to B deviates but slightly from the straight line DB. Hence

$$t = \int_A^B dt = \int_A^B x d\phi = \frac{1}{EI}\int_A^B Mxdx \qquad (2)$$

The distance $DB = t$ is called the *tangential deviation;* since it represents the distance through which the point B has been displaced by the curvature of the beam, when AD is assumed as the original position.

In Eqs. (1) and (2), I is the *gross* moment of inertia of the cross-section. No deductions are made for holes, as is the case when the strength of a beam is being computed.

The expression $Mxdx$ represents the moment of the elemental area Mdx about the ordinate through B. Hence the integral expression $\int_A^B Mxdx$ represents the moment of the area $QPRS$ about the ordinate through B, and is called the area-moment of $QPRS$ about B. The area-moment is expressed in inches^3-pounds, when M is expressed in inch-pounds. Since EI is expressed in inches^2-pounds, the tangential deviation t is expressed in inches. The second principle may now be stated.

139. Second Principle.—*If the tangent to the elastic curve is drawn through any point A, the tangential deviation at any other point B may be obtained by finding the area of the* M*-diagram between ordinates through A and B; and dividing by EI the moment of this area about the ordinate through B.*

Let J represent the centroid of the area $QPRS$, and let k be the distance from J to the ordinate through B. Then

$$t = \frac{\text{area } QPRS}{EI} k$$

Since, from the first principle

$$\frac{\text{area } QPRS}{EI} = \phi$$

then $$t = k\phi$$

Hence the tangents to the elastic curve at any two points A and B intersect on the ordinate through the centroid of the M-diagram included between the ordinates through A and B.

SEC. II. SIMPLE BEAMS OF UNIFORM CROSS-SECTION

140. The beam in Fig. 129*a* is a 2 by 1 in. piece of wood laid

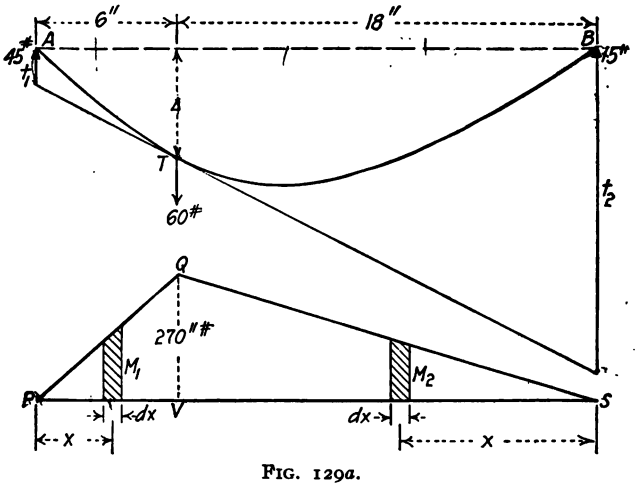


FIG. 129*a*.

flatwise. $I = \frac{1}{6}$ in.^{4} $E =$ 1,500,000 lb./in.^{2} Hence $EI =$ 250,000 in.^{2}-lb. The M-diagram is PQS. The deflection Δ under the load will be determined in several ways, by drawing

the tangent to the elastic curve through different points as shown in Figs. 129*a-b-c*. Considerable time and labor may be saved by exercising good judgment in choosing the most advantageous point in the elastic curve through which the tangent is to be drawn.

In Fig. 129*a* the tangent to the elastic curve ATB is drawn through T. The deflection Δ is quickly found after the tangential deviations t_1 and t_2 have been computed.

$$t_1 = \frac{1}{EI}\int_0^6 M_1 x dx$$

Where M_1 is the bending moment at any distance x from the ordinate, on which the tangential deviation is required. Hence $M_1 = 45x$.

$$t_1 = \frac{1}{EI}\int_0^6 45\, x^2 dx = \frac{3{,}240}{EI}$$

The origin for t_2 is at B, hence $M_2 = 15x$, and

$$t_2 = \frac{1}{EI}\int_0^{18} M_2 x dx = \frac{29{,}160}{EI}$$

$$\Delta = t_1 + \frac{6}{24}(t_2 - t_1) = \frac{9{,}720}{EI} = \frac{9{,}720}{1{,}500{,}000} = 0.039 \text{ in.}$$

The expression $\int_0^6 M_1 x dx$ represents the area-moment of PQV about P, hence

$$t_1 = \frac{1}{EI}(270 \times 3 \times 4) = \frac{3{,}240}{EI}$$

Likewise, the expression $\int_0^{18} M_2 x dx$ represents the area-moment of SQV about S, hence

$$t_2 = \frac{1}{EI}(270 \times 9 \times 12) = \frac{29{,}160}{EI}$$

Thus it is clear that, when the M-diagram can be conveniently divided into portions whose areas and centroids are easily found, a semigraphic or geometric solution can be quickly made. The area of the M-diagram to be considered in each case is included between two ordinates. One ordinate passes through the point of tangency, on the other ordinate the tangential deviation is found; and the moment of this area is taken about the latter ordinate.

In Fig. 129*b* the tangent is drawn through A. The area-

moment for t_4 is PQS about S; and for t_3, the area-moment is PQV about QV.

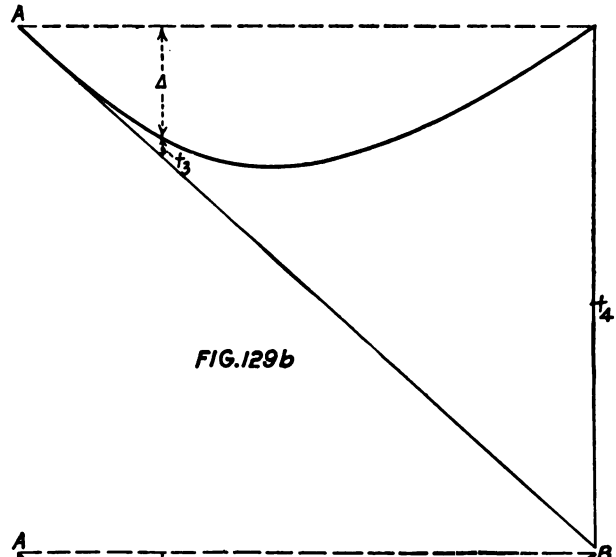


FIG.129b

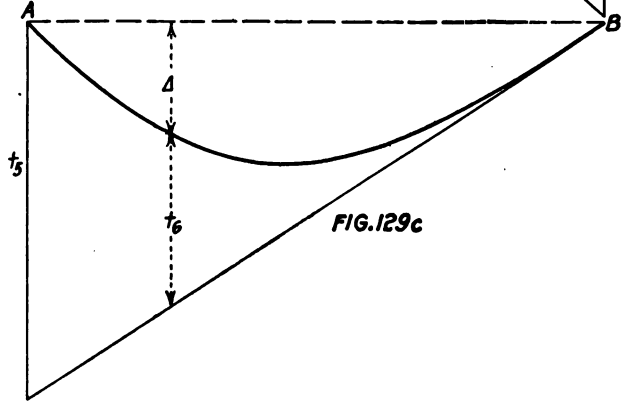


FIG.129c

$$t_4 = \frac{1}{EI}\begin{bmatrix} 270 \times 9 \times 12 = 29{,}160 \\ 270 \times 3(18 + 2) = 16{,}200 \end{bmatrix} = \frac{45{,}360}{EI}$$

$$t_3 = \frac{1}{EI}(270 \times 3 \times 2) = \frac{1{,}620}{EI}$$

$$\Delta + t_3 = \frac{t_4}{4} = \frac{11{,}340}{EI}$$

$$\Delta = \frac{11{,}340 - 1{,}620}{EI} = \frac{9{,}720}{EI} \textit{ as before.}$$

In the algebraic solution, the origin for t_4 is at S. $M = 15x$ for values of x between 0 and 18, and $M = 15x - 60(x - 18) = 1{,}080 - 45x$, for values of x between 18 and 24, hence

$$t_4 = \frac{1}{EI}\int_0^{24} Mx\,dx = \frac{1}{EI}\int_0^{18} 15x^2\,dx + \frac{1}{EI}\int_{18}^{24} (1{,}080x - 45x^2)\,dx$$
$$= \frac{29{,}160 + 16{,}200}{EI} = \frac{45{,}360}{EI}$$

The origin for t_3 is at V, hence $M = 45(6 - x)$, and

$$t_3 = \frac{45}{EI}\int_0^6 (6x - x^2)\,dx = \frac{1{,}620}{EI}$$

The geometric solution is considerably shorter when M is not a continuous function of x as in the case of t_4.

In Fig. 129*c* the tangent is drawn through B.

$$t_5 = \frac{1}{EI}\begin{bmatrix} 270 \times 3 \times 4 = & 3{,}240 \\ 270 \times 9(6 + 6) = & 29{,}160 \end{bmatrix} = \frac{32{,}400}{EI}$$

$$t_6 = \frac{1}{EI}(270 \times 9 \times 6) = \frac{14{,}580}{EI}$$

$$\Delta + t_6 = \frac{3}{4} t_5 = \frac{24{,}300}{EI}$$

$$\Delta = \frac{24{,}300 - 14{,}580}{EI} = \frac{9{,}720}{EI} \textit{ as before.}$$

141. Maximum Deflection.—Let X (Fig. 129*d*) represent the point of maximum deflection. Since the tangent through X is horizontal, $\Delta max = t_7 = t_8$. Let KL be the ordinate in the M-diagram at the point of maximum deflection, and let $LS = a$. Then $KL = 15a$. Let ϕ represent the angle which the tangent through B makes with the horizontal tangent through X, then

$$\angle ABC = \angle BID = \phi$$

and $$24\phi = t_5$$

or $$\phi = \frac{t_5}{24} = \frac{1{,}350}{EI}$$

also $$\phi = \frac{\text{area } KLS}{EI} = \frac{7.5a^2}{EI}$$

hence $$7.5a^2 = 1{,}350$$
$$a = 13.42$$

Since the centroid of the triangular area KLS is on the ordinate through I,

$$ID = \frac{2}{3}a$$

$$t_8 = \frac{2}{3}a\phi = \left(\frac{2a}{3}\right)\left(\frac{1{,}350}{EI}\right) = \frac{900a}{EI} = \frac{12{,}078}{EI} = 0.048 \text{ in.}$$

or $$t_8 = \left[\frac{\text{area-moment of}}{KLS \text{ about } S}\right] \div EI = \frac{5a^3}{EI} = \frac{12{,}078}{EI}$$

The distance a might also be found by equating the values of t_7 and t_8 without any reference to the tangent through B.

The general expression will be developed for the deflection of a simple beam l in. long when supporting a single concen-

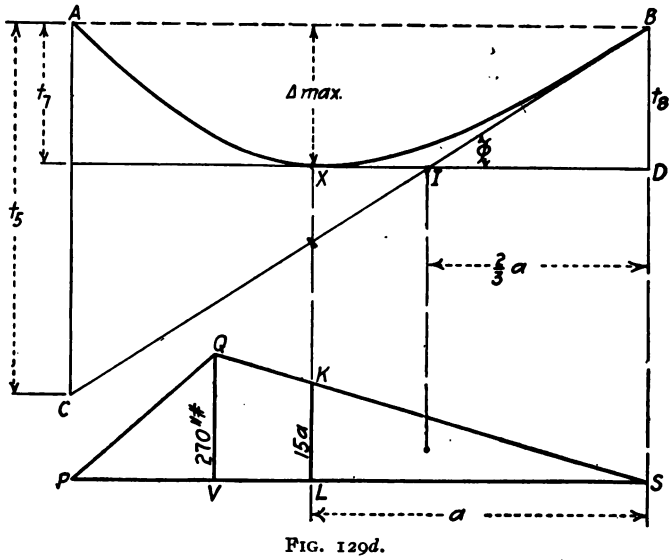


FIG. 129d.

trated load of P lb. at any distance kl from the left support, (Figs. 130 and 131). The deflection Δ is found at T, a distance cl from the left support.

When $c < k$.

In Fig. 130 the tangent CD is drawn through T.

$$EIt_1 = c(1 - k)Pl\left(c\,\frac{l}{2}\right)\left(\frac{2}{3}\,cl\right)$$

EIt_2 equals the area-moment of $KQSL$ about S which is the

area-moment of PQS about S, minus the area-moment of PKL about S.

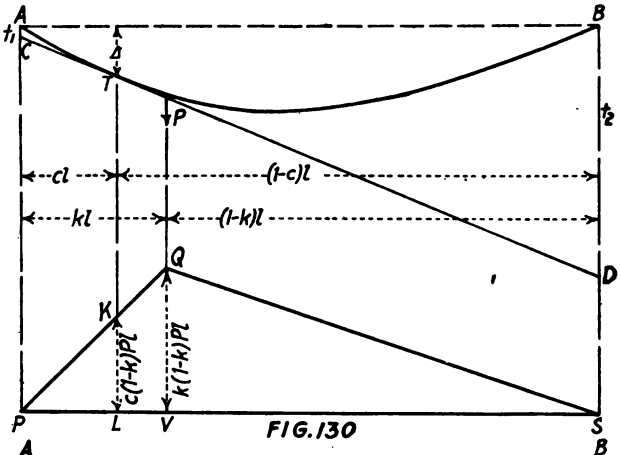


FIG. 130

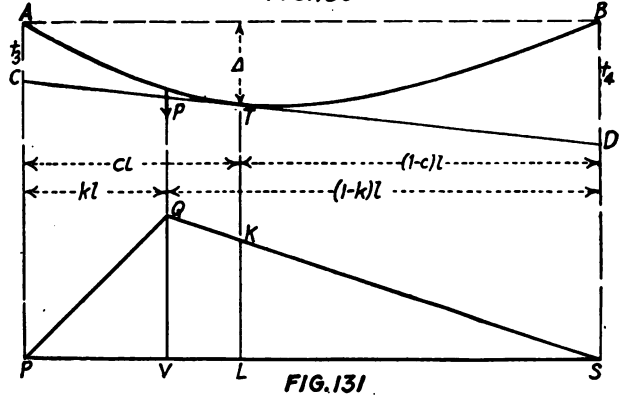


FIG. 131

$$EIt_2 = k(1-k)Pl\left(\frac{l}{2}\right)\frac{2}{3}\left(l-\frac{kl}{2}\right) - c(1-k)Pl\left(c\frac{l}{2}\right)\left(l-\frac{2}{3}cl\right)$$

$$t_1 = \frac{Pl^3}{6EI}(1-k)2c^3$$

$$t_2 = \frac{Pl^3}{6EI}(1-k)(2k-k^2-3c^2+2c^3)$$

$$\Delta = t_1 + c(t_2 - t_1)$$

$$\Delta = \frac{Pl^3}{6EI}(2ck - 3ck^2 - c^3 + ck^3 + c^3k) \qquad (3)$$

When $c > k$.

In Fig. 131 the tangent CD is drawn through T as before.

$$t_3 = \frac{Pl^3}{6EI} k(3c^2 - 2c^3 - k^2)$$

$$t_4 = \frac{Pl^3}{6EI} 2k(1 - c)^3$$

$$\Delta = t_3 + c(t_4 - t_3)$$

$$\Delta = \frac{Pl^3}{6EI}(2ck - 3c^2k - k^3 + c^3k + ck^3) \qquad (4)$$

The deflection at the load may be obtained from either Eq. (3) or (4). Since $c = k$ for this condition, either equation reduces to

$$\Delta = \frac{Pl^3}{3EI} k^2(1 - k)^2$$

Let F represent the expression in the parenthesis of equation (3) when c is less than k, and the expression in the parenthesis of Eq. (4) when c is greater than k, then in general

$$\Delta = \frac{Pl^3}{6EI} F \qquad (5)$$

The values of F for various values of c and k are given in Table I; or may be found from Fig. 132. In Eq. (5) l is the length of the beam in inches, and P is the load in pounds at any distance kl from the nearer support. Δ is the deflection in inches at any distance cl from the same support. E is the modulus of elasticity in pounds per square inch, and I is the moment of inertia of the constant cross-section of the beam about the neutral axis, measured in inches^4.

142. Point of Maximum Deflection.—The maximum deflection occurs in the longer segment of Fig. 131 where c is greater than k, and at the point where the tangent through T is horizontal; hence the value of c for $\Delta_{max.}$ may be found by equating the expressions for t_3 and t_4, whence

$$c = 1 - \sqrt{(1 - k^2)/3}$$

The values of c and the corresponding values of F for $\Delta_{max.}$ are also given in Table I.

Since the limits of k are 0 and 0.5, all values of c for maximum deflection will fall between 0.4227 and 0.5. Hence the point of maximum deflection for a single load is between the load

and the center of the span, and always relatively near the center. The most eccentric loading which a simple beam of uniform cross-section and span l can experience, occurs when a

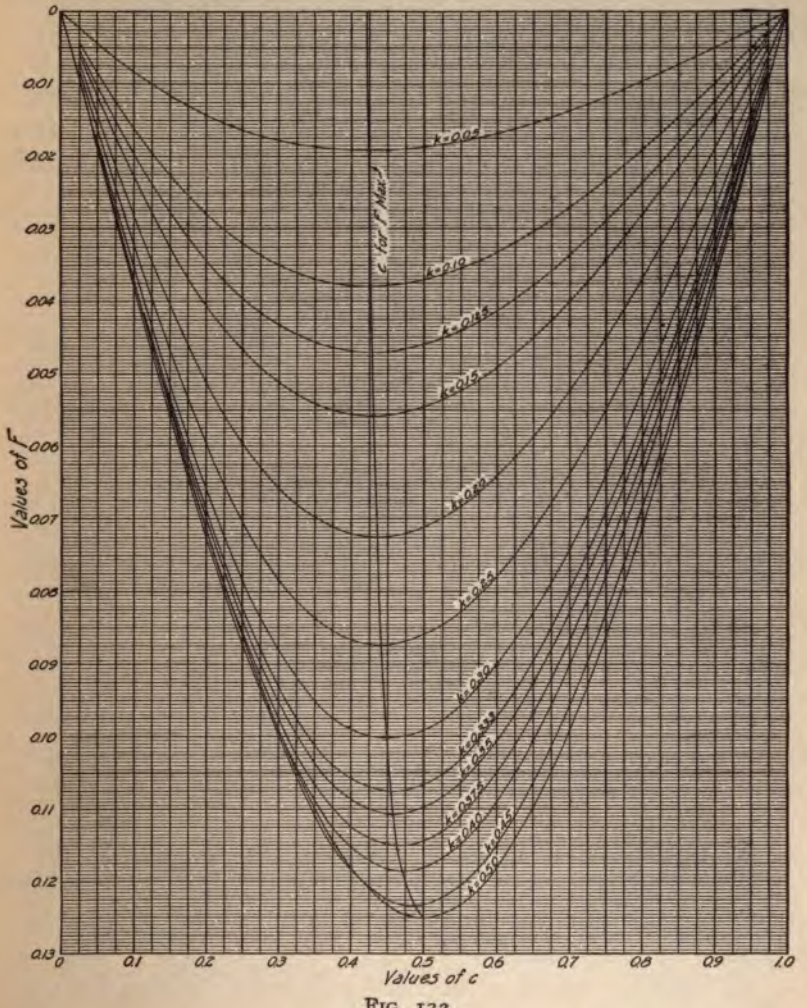


FIG. 132.

single load is adjacent to one of the supports, and k is on the point of becoming zero. Under this condition the point of maximum deflection cannot be at a distance greater than $0.0773l$ from the center of the span. Any second load applied to the

beam must necessarily throw the point of maximum deflection nearer the center. Hence the point of maximum deflection of a simple beam of uniform cross-section, loaded in any manner, will be near the center and not more than 0.0773 of its length from the center.

143. A 20-in. 65-lb. I-beam supports two loads of 30,000 lb. each (Fig. 133). Since the loads are symmetrically placed, the elastic curve and M-diagram are symmetrical about the center. The tangent to the elastic curve at the center, drawn

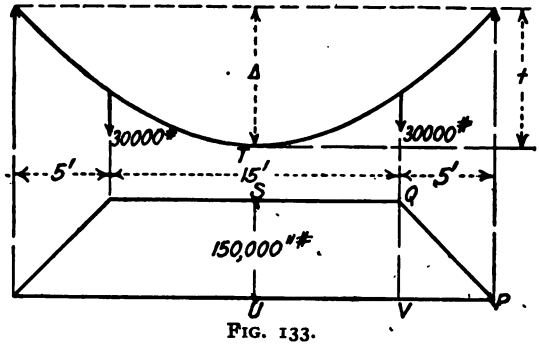


FIG. 133.

through T, is horizontal and $\Delta = t$. $E = 29{,}000{,}000$ lb./in^2., $I = 1{,}169.5$ in^4.; hence $EI = 33{,}915{,}500{,}000$ in^2.-lb.

Area-moment of $PQSU$ about P.

area PQV $150{,}000 \times 2.5 \times 3.33 = 1{,}250{,}000$

area $QSUV$ $150{,}000 \times 7.5 \times 8.75 = \underline{9{,}843{,}750}$

$11{,}093{,}750$ ft^3.-lb.

$$\Delta = t = \frac{11{,}093{,}750 \times 1{,}728}{33{,}915{,}500{,}000} = 0.565 \text{ in.}$$

When the length of a beam is expressed in feet, and the loads are expressed in pounds, the area-moment will be expressed in foot^3-pounds; and the factor 1,728 is introduced if EI is expressed in inch^2-pounds.

EI may be expressed in foot^2-pounds by dividing by 144, whence

$EI = 235{,}521$ ft.^{2}-lb., then

$$\Delta = t = \frac{11{,}093{,}750 \text{ ft.}^3\text{-lb.}}{235{,}521 \text{ ft.}^2\text{-lb.}} = 0.0471 \text{ ft.} = 0.565 \text{ in.}$$

The deflection at the center may be found from Table I.

$k = 0.2$ and $c = 0.5$; therefore $F = 0.071$ for each load, then

$$\Delta = \frac{2Pl^3}{6EI} \times 0.071 = \frac{60{,}000(25 \times 12)^3 0.071}{6 \times 33{,}915{,}500{,}000} = 0.565 \text{ in.}$$

144. Deflection Under Uniform Load.—The beam in Fig. 134 supports a uniform load and the M-diagram PQS is a

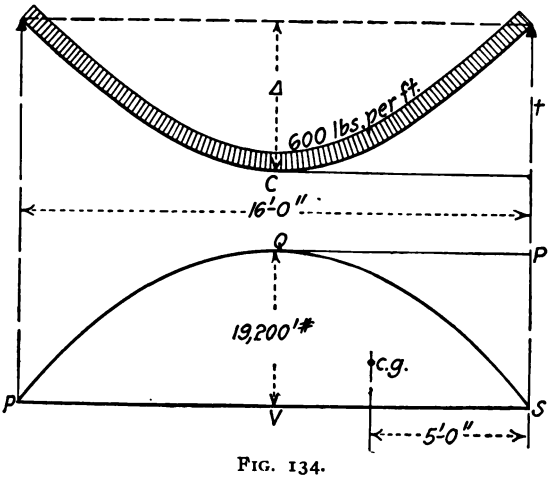


FIG. 134.

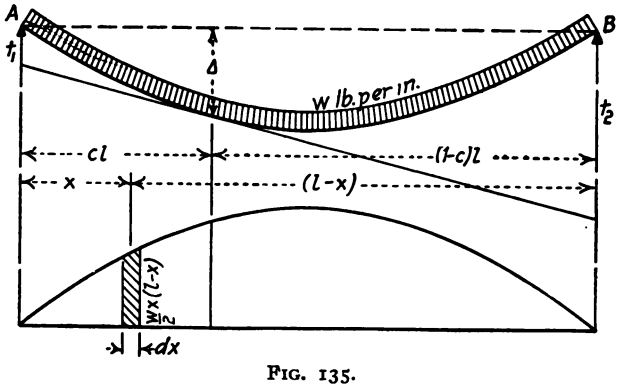


FIG. 135.

parabola. The maximum deflection is at the center of the span. The tangent to the elastic curve at C is horizontal, and $\Delta = t$. EIt = the area-moment of SQV about S. The area SQV is two-thirds the area of the rectangle $QPSV$, and the centroid of the area SQV is five-eighths of VS, hence

$$\Delta = t = \frac{1}{EI}(19{,}200 \times 8 \times \tfrac{2}{3} \times 5 \times 1{,}728) = \frac{884{,}736{,}000}{EI}$$

TABLE I.—VALUES OF F

c \ k	0.05	0.10	0.125	0.15	0.20	0.25	0.30	0.333	0.35	0.375	0.40	0.45	0.50
0.05	0.0045	0.0084	0.0101	0.0117	0.0143	0.0163	0.0178	0.0184	0.0187	0.0190	0.0191	0.0191	0.0187
0.10	0.0084	0.0162	0.0196	0.0227	0.0280	0.0321	0.0350	0.0364	0.0369	0.0375	0.0378	0.0378	0.0370
0.15	0.0117	0.0227	0.0278	0.0325	0.0405	0.0467	0.0512	0.0533	0.0541	0.0550	0.0556	0.0557	0.0546
0.20	0.0143	0.0280	0.0344	0.0405	0.0512	0.0596	0.0658	0.0687	0.0699	0.0712	0.0720	0.0723	0.0710
0.25	0.0163	0.0320	0.0396	0.0467	0.0596	0.0703	0.0783	0.0822	0.0837	0.0855	0.0866	0.0873	0.0859
0.30	0.0178	0.0350	0.0433	0.0512	0.0658	0.0783	0.0882	0.0931	0.0951	0.0974	0.0990	0.0999	0.0990
0.35	0.0187	0.0369	0.0457	0.0541	0.0699	0.0837	0.0951	0.1011	0.1035	0.1065	0.1087	0.1107	0.1098
0.40	0.0191	0.0378	0.0468	0.0556	0.0720	0.0866	0.0990	0.1058	0.1087	0.1124	0.1152	0.1183	0.1180
0.45	0.0191	0.0378	0.0469	0.0557	0.0723	0.0873	0.1002	0.1075	0.1107	0.1149	0.1183	0.1225	0.1232
0.50	0.0187	0.0370	0.0459	0.0546	0.0710	0.0859	0.0990	0.1065	0.1098	0.1143	0.1180	0.1232	0.1250
0.55	0.0179	0.0354	0.0440	0.0523	0.0682	0.0827	0.0955	0.1030	0.1063	0.1108	0.1148	0.1205	0.1232
0.60	0.0168	0.0332	0.0412	0.0491	0.0640	0.0778	0.0900	0.0972	0.1005	0.1049	0.1088	0.1148	0.1180
0.65	0.0153	0.0304	0.0377	0.0449	0.0586	0.0713	0.0827	0.0894	0.0925	0.0967	0.1005	0.1063	0.1098
0.70	0.0136	0.0270	0.0335	0.0399	0.0522	0.0636	0.0738	0.0799	0.0827	0.0866	0.0900	0.0955	0.0990
0.75	0.0117	0.0232	0.0288	0.0343	0.0449	0.0547	0.0636	0.0689	0.0713	0.0747	0.0778	0.0827	0.0859
0.80	0.0096	0.0190	0.0236	0.0281	0.0368	0.0449	0.0522	0.0566	0.0586	0.0615	0.0640	0.0682	0.0710
0.85	0.0073	0.0145	0.0180	0.0215	0.0281	0.0343	0.0399	0.0433	0.0449	0.0471	0.0491	0.0523	0.0546
0.90	0.0049	0.0098	0.0122	0.0145	0.0190	0.0232	0.0270	0.0293	0.0304	0.0319	0.0332	0.0354	0.0370
0.95	0.0025	0.0049	0.0061	0.0073	0.0096	0.0117	0.0136	0.0148	0.0153	0.0161	0.0168	0.0179	0.0187

For Maximum Deflection

c	0.4234	0.4255	0.4272	0.4292	0.4343	0.4409	0.4492	0.4557	0.4592	0.4648	0.4708	0.4850	0.5000
F	0.0192	0.0379	0.0470	0.0558	0.0724	0.0873	0.1002	0.1075	0.1107	0.1150	0.1185	0.1234	0.1250

In Fig. 135 the span is l in., and the uniform load is w lb. per inch. The deflection at any distance cl from A will be found.

At any distance x from either support, $M = \frac{\omega}{2} x\,(l - x)$.

$$t_1 = \frac{1}{EI}\int_0^{cl} Mx\,dx = \frac{\omega l^4}{24}(4c^3 - 3c^4)$$

$$t_2 = \frac{1}{EI}\int_0^{(1-c)l} Mx\,dx = \frac{\omega l^4}{24}(1 - 6c^2 + 8c^3 - 3c^4)$$

$$\Delta = t_1 + c(t_2 - t_1)$$

$$\Delta = \frac{wl^4}{24EI}(c - 2c^3 + c^4)$$

Let W = the total uniform load, then $W = wl$, whence

$$\Delta = \frac{Wl^3}{24EI}(c - 2c^3 + c^4) \tag{6}$$

Let J represent the expression in the parenthesis of Eq. (6) then

$$\Delta = \frac{Wl^3}{24EI} J \tag{7}$$

The values of J for various values of c are given in Table II; in which l is the length of the beam in inches, W is the total uniformly distributed load in pounds and Δ is the deflection in inches at any distance cl from the nearer support. E is the modulus of elasticity in pounds per square inch, and I is the moment of inertia of the constant cross-section of the beam about the neutral axis, expressed in inches.[4]

For $\Delta_{max.}$, $c = 0.5$, hence

$$\Delta_{max.} = \frac{Wl^3}{24EI} \times 0.3125 = \frac{5Wl^3}{384EI}$$

TABLE II

c	J	c	J
0.05	0.0498	0.30	0.2541
0.10	0.0981	0.35	0.2793
0.15	0.1438	0.40	0.2976
0.20	0.1856	0.45	0.3088
0.25	0.2227	0.50	0.3125

145. Deflection for Load of Uniformly Varying Intensity.—The beam in Fig. 136 supports a load of uniformly varying intensity. The total load is W lb., and the length of the beam is l in. The bending moment at any distance x from A is

$$M_1 = \frac{W}{3l^2}(l^2x - x^3)$$

The bending moment at any distance x from B is

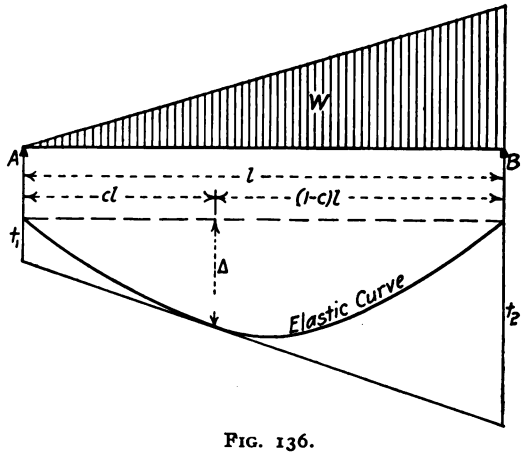


FIG. 136.

$$M_2 = \frac{W}{3l^2}(2l^2x - 3lx^2 + x^3)$$

$$t_1 = \frac{1}{EI}\int_0^{cl} M_1 x dx = \frac{Wl^3}{180EI}(20c^3 - 12c^5)$$

$$t_2 = \frac{1}{EI}\int_0^{(1-c)l} M_2 x dx = \frac{Wl^3}{180EI}(7 - 30c^2 + 20c^3 + 15c^4 - 12c^5)$$

$$\Delta = t_1 + c(t_2 - t_1) = \frac{Wl^3}{180EI}(7c - 10c^3 + 3c^5)$$

The value of c for $\Delta_{max.}$ may be found by equating t_1 and t_2, whence

$$15c^4 - 30c^2 = -7$$

$$c = 0.519$$

$$\Delta_{max.} = \frac{Wl^3}{180EI} \times 2.348 = \frac{Wl^3}{0.01311EI}$$

The intensity of the load in Fig. 137 increases uniformly from each support to the center of the span. The total load is

W lb., and the length of the beam is l in. The bending moment at any distance x between the end and center is

$$M_1 = \frac{W}{6l^2}(3l^2x - 4x^3)$$

The bending moment at any distance x, when x is greater than $\frac{1}{2}l$ is

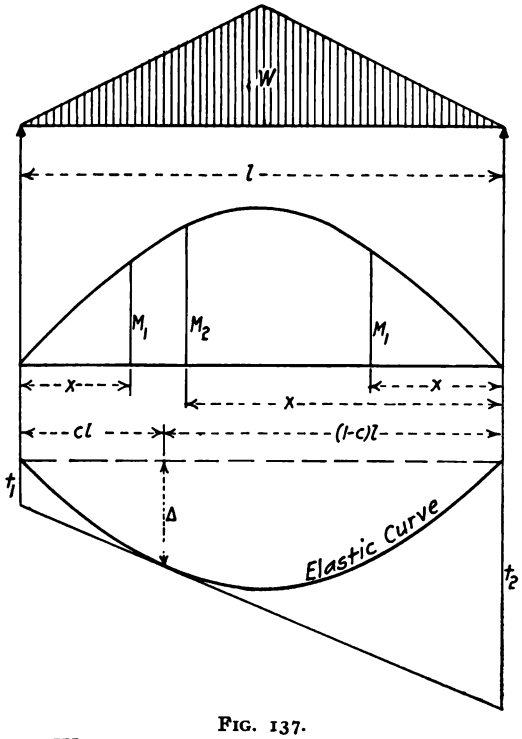


FIG. 137.

$$M_2 = \frac{W}{6l^2}(-l^3 + 9l^2x - 12lx^2 + 4x^3)$$

$$t_1 = \frac{1}{EI}\int_0^{cl} M_1 x\,dx = \frac{Wl^3}{60EI}(10c^3 - 8c^5)$$

$$t_2 = \frac{1}{EI}\int_0^{\frac{l}{2}} M_1 x\,dx + \frac{1}{EI}\int_{\frac{l}{2}}^{(1-c)l} M_2 x\,dx$$

$$= \frac{Wl^3}{60EI}\left(\frac{25}{8} - 15c^2 + 10c^3 + 10c^3 - 8c^5\right)$$

$$\Delta = t_1 + c(t_2 - t_1) = \frac{Wl^3}{480IE}(25c - 40c^3 + 16c^5)$$

in which c may have any value between 0 and $\frac{1}{2}$. The value of c for $\Delta_{max.}$ may be found by equating t_1 and t_2 whence,

$$10c^4 - 15c^2 = -\frac{25}{8}$$

$$c = \frac{1}{2}$$

$$\Delta_{max.} = \frac{Wl^3}{60EI}$$

146. Deflection for Several Concentrated Loads.—A 24-in. by 80-lb. I-beam supports three loads (Fig. 138). The linear

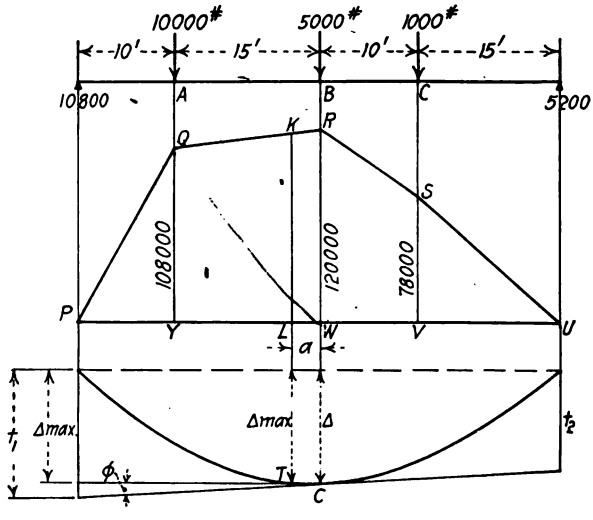


FIG. 138.

dimensions of the beam are expressed in feet and the units in which E and I are usually expressed will be changed accordingly. $I = 2{,}087$ in.^{4} $E = 29{,}000{,}000$ lb./in.^{2} Hence $EI = 60{,}532{,}000{,}000$ in.^{2}-lb. or $420{,}400{,}000$ ft.^{2}-lb. The tangent is drawn through C at the center of the span. The deflection due to the weight of the beam will be considered later.

Area-moment about P.

area PQY	$108{,}000 \times 5 \times 6.67$	$= 3{,}600{,}000$
area YQW	$108{,}000 \times 7.5 \times 15$	$= 12{,}150{,}000$
area QRW	$120{,}000 \times 7.5 \times 20$	$= 18{,}000{,}000$
		$33{,}750{,}000$ ft.^{3}-lb.

$$t_1 = \frac{33{,}750{,}000}{420{,}400{,}000} = 0.0803 \text{ ft.}$$

Area-moment about U.

area USV $78{,}000 \times 7.5 \times 10 = 5{,}850{,}000$
area SVW $78{,}000 \times 7.5 \times 18.33 = 7{,}150{,}000$
area SRW $120{,}000 \times 5 \times 21.67 = 13{,}000{,}000$
$26{,}000{,}000$ ft.^{3}-lb.

$$t_2 = \frac{26{,}000{,}000}{420{,}400{,}000} = 0.0618 \text{ ft.}$$

$$\Delta = \frac{t_1 + t_2}{2} = 0.071 \text{ ft.} = 0.85 \text{ in.}$$

The maximum deflection caused by the three loads is at T, where the tangent is horizontal, and the ordinate in the M-diagram is KL. Let $LW = a$ and let ϕ be the angle made by the two tangents; then ϕ represents the slope of the tangent through C. The beam is 50 ft. long

hence $$\phi = \frac{t_1 - t_2}{50} = \frac{0.0185}{50} = 0.00037$$

also $$\phi = \frac{\text{area } KRWL}{EI}$$

hence area $KRWL = \phi\, EI = 155{,}000$ ft.^{2}-lb.

whence $a = 1.3$ ft.

The centroid of the area $KRWL$ is approximately 24.35 ft. from P, and the area-moment of $KRWL$ about P is

$$155{,}000 \times 24.35 = 3{,}774{,}000 \text{ ft.}^3\text{-lb.}$$

$$\Delta_{max.} = t_1 - \frac{3{,}774{,}000}{EI} = \frac{29{,}976{,}000}{EI}$$

$$\Delta_{max.} = \frac{29{,}976{,}000}{420{,}400{,}000} = 0.0713 \text{ ft.} = 0.86 \text{ in.}$$

Although the loads are eccentric, it is clear that there is practically no difference between the deflection at the center and the maximum deflection.

The deflection at the center may be found from Eq. 5, page 218. The coefficients F are given in Table I. $c = 0.5$; $k =$

0.2 for the load at A; 0.5, for the load at B; and 0.3 for the load at C; hence

$$\frac{10{,}000 \times 50^3}{6} \times 0.071 = 14{,}792{,}000$$

$$\frac{5{,}000 \times 50^3}{6} \times 0.125 = 13{,}021{,}000$$

$$\frac{1{,}000 \times 50^3}{6} \times 0.099 = \frac{2{,}062{,}000}{29{,}875{,}000}\ \text{ft}^3\text{.-lb.}$$

$$\Delta = \frac{29{,}875{,}000}{420{,}400{,}000} = 0.071 \text{ ft.} = 0.85 \text{ in.}$$

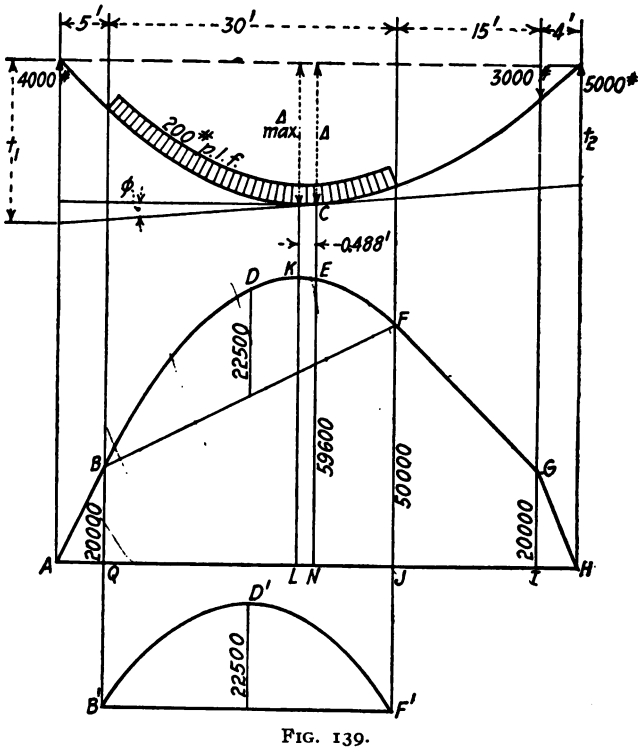


FIG. 139.

The deflection at the center, due to the weight of the beam, may be found from equation 7, page 223. The coefficient of J when $c = 0.5$ is 0.3125. $W = 50 \times 80 = 4{,}000$.

$$\Delta = \frac{4{,}000 \times 50^3}{24 \times 420{,}400{,}000} \times 0.3125 = 0.0155 \text{ ft.} = 0.19 \text{ in.}$$

The total deflection at the center is 0.85 + 0.19 = 1.04 in., which in this case may be assumed without appreciable error as the maximum deflection. A deflection of $\frac{1}{360}$ of the span is considered not excessive.

147. Deflection for Uniform and Concentrated Loads.— In Fig. 139, the tangent is drawn through C at the center of the span. Assume that EI is expressed in foot^2-pounds. The M-diagram under the uniform load cannot be accurately divided into triangles, and an integration is necessary if an accurate solution is desired. A sufficiently accurate solution for all practical purposes may be obtained by the geometric process by dividing the area $QBDFJ$ by vertical ordinates into strips, so narrow that their areas may be considered trapezoidal. The accurate method by integration is given below. Let M_1 represent the bending moment under the uniform load at any distance x from the left support; and M_2, the bending moment under the uniform load at any distance x from the right support; then

$$M_1 = -100x^2 + 5{,}000x - 2{,}500$$

and

$$M_2 = -100x^2 + 5{,}800x - 24{,}100$$

Area-moment about A

area ABQ	$20{,}000 \times 2.5 \times 3.33$ =	166,667
area $BENQ$	$\int_5^{27} M_1 x\,dx$ =	18,446,266
		18,612,933 ft^3.-lb.

$$t_1 = \frac{18{,}612{,}933}{EI} \text{ ft.}$$

Area-moment about H

area GHI	$20{,}000 \times 2 \times 2.67$ =	106,667
area GIJ	$20{,}000 \times 7.5 \times 9$ =	1,350,000
area FGJ	$50{,}000 \times 7.5 \times 14$ =	5,250,000
area $FENJ$	$\int_{19}^{27} M_2 x\,dx$ =	10,332,600
		17,039,267 ft.^{3}-lb.

$$t_2 = \frac{17{,}039{,}267}{EI} \text{ ft.}$$

The deflection at the center is

$$\Delta = \frac{t_1 + t_2}{2} = \frac{17{,}826{,}100}{EI} \text{ ft.}$$

The slope of the tangent is

$$\phi = \frac{t_1 - t_2}{54} = \frac{29{,}142}{EI}$$

Let KL represent the ordinate in the M-diagram at the point of maximum deflection, then

$$\phi = \frac{\text{area } KENL}{EI}$$

therefore area $KENL = 29{,}142$
whence $LN = 0.488$ ft.

The area-moment of $KENL$ about A is

$$29{,}142 \times 26.756 = 779{,}723$$

$$\Delta_{max.} = t_1 - \frac{779{,}723}{EI} = \frac{17{,}833{,}211}{EI}$$

The area-moment of $KENL$ about H is

$$29{,}142 \times 27.244 = 793{,}945$$

$$\Delta_{max.} = t_2 + \frac{793{,}945}{EI} = \frac{17{,}833{,}212}{EI}$$

When the tangent is drawn to the elastic curve at the right end of the uniform load, the tangential deviations t_3 and t_4, at the left and right supports respectively, may be determined by the geometric process; for if a straight line be drawn from B to F, the area of the M-diagram BDF has all the properties of the M-diagram $B'D'F'$ for a beam 30 ft. long when uniformly loaded with 200 lb. per foot (Fig. 139*b*). See Article **60.**

Area-moment about A

area ABQ	$20{,}000 \times 2.5 \times 3.33$	$= 166{,}667$
area BQJ	$20{,}000 \times 15 \times 15$	$= 4{,}500{,}000$
area FBJ	$50{,}000 \times 15 \times 25$	$= 18{,}750{,}000$
area BDF	$22{,}500 \times 30 \times \frac{2}{3} \times 20$	$= 9{,}000{,}000$
		$32{,}416{,}667$ ft³.-lb.

Area-moment about H

area GHI	$20{,}000 \times 2 \times 2.67$	$= 106{,}667$
area GIJ	$20{,}000 \times 7.5 \times 9$	$= 1{,}350{,}000$
area FGJ	$50{,}000 \times 7.5 \times 14$	$= 5{,}250{,}000$
		$6{,}706{,}667$ ft³.-lb.

The slope of the tangent is

$$\phi = \frac{t_3 - t_4}{54} = \frac{476{,}315}{EI} = \frac{\text{area } KFJL}{EI}$$

The area $EFJN$, when considered as four trapezoidal areas, each 2 ft. wide, is 446,440; hence the approximate area of $KENL$ is

$$476{,}315 - 446{,}400 = 29{,}915$$

from which we find that the ordinate KL is located about 0.5 ft. to the left of the center of the beam as before.

SEC. III. MAXWELL'S THEOREM OF RECIPROCAL DISPLACEMENTS

148. Maxwell's Theorem of Reciprocal Displacements establishes a mutual relation between any two points in a structure. This theorem, when considered in connection with the deflection of beams, may be stated as follows: If the load P at A

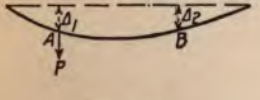


FIG. 140.

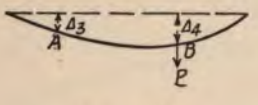


FIG. 141.

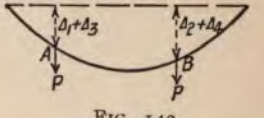


FIG. 142.

(Fig. 140) causes a deflection Δ_2 at B, and the load P at B (Fig. 141) causes a deflection Δ_3 at A; then, according to Maxwell's theorem, $\Delta_2 = \Delta_3$. Let Figs. 140, 141 and 142 represent the deflections of a beam when the loads are applied gradually. When A (Fig. 140) has received its full load, the work done is $\frac{1}{2}P\Delta_1$. With a full load P at A, let another load P be gradually added at B. The deflections as shown in Fig. 142 will result. The point A, with the full load P, moves through the additional distance Δ_3; and the point B moves through the additional distance Δ_4, as the load P is gradually applied at B. Hence the total work done is

$$\tfrac{1}{2}P\Delta_1 + P\Delta_3 + \tfrac{1}{2}P\Delta_4$$

If B is loaded first and then A is loaded, the total work done is

$$\tfrac{1}{2}P\Delta_4 + P\Delta_2 + \tfrac{1}{2}P\Delta_1$$

The total amount of work done in each case is the same, hence

$$\Delta_2 = \Delta_3$$

Maxwell's law may be verified by Table I. When A and B are on the same side of the center, the values of k and c for Fig. 140 become interchanged for Fig. 141. For example, $F = 0.0658$ when $k = 0.2$ and $c = 0.3$; likewise, $F = 0.0658$ when $k = 0.3$ and $c = 0.2$. When A and B are on opposite sides of the center the application is made as follows: In Fig. 140 let $k = 0.3$ and $c = 0.8$; then in Fig. 141, $k = 0.2$ and $c = 0.7$; whence $F = 0.0522$ in each case. Maxwell's law renders excellent service in the solution of statically indeterminate structures.

SEC. IV. CANTILEVER BEAMS

149. The beam in Fig. 143 supports a single load P at the free end. It is fixed in the wall at A in such a way that the

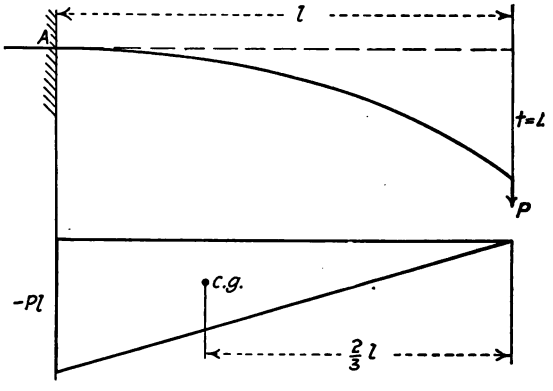


FIG. 143.

tangent to the elastic curve at A remains horizontal, hence the deflection at the free end is

$$\Delta = t = \frac{1}{EI}(-Pl)\left(\frac{l}{2}\right)\left(\frac{2}{3}l\right) = -\frac{Pl^3}{3EI}$$

The negative sign indicates that the elastic curve deviates below the tangent.

The beam in Fig. 144 supports the load W uniformly distributed. The M-diagram is SQV. The curve SV is a para-

bola with the vertex at S, hence the deflection at the free end is

$$\Delta = t = \frac{1}{EI}\left(-\frac{Wl}{2}\right)\left(\frac{l}{3}\right)\left(\frac{3}{4}l\right) = -\frac{Wl^3}{8EI}$$

A 7-in., 15-lb. I-beam (Fig. 145) supports a load of 2,000 lb.

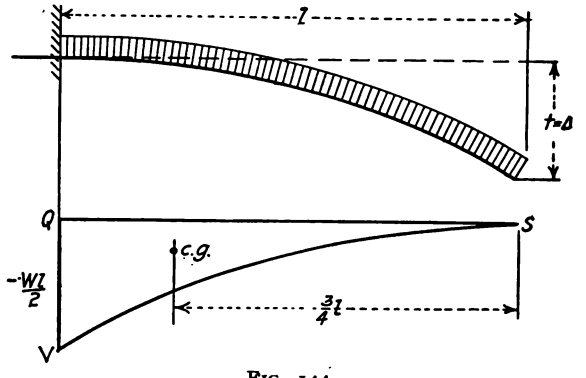


FIG. 144.

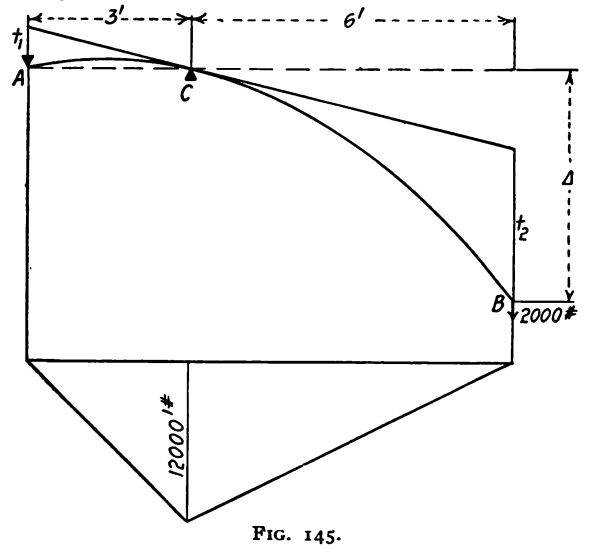


FIG. 145.

EI = 1,050,000,000 in.^{2}-lb. The tangent is drawn through C.

$$t_1 = \frac{-12{,}000 \times 1.5 \times 2 \times 1{,}728}{1{,}050{,}000{,}000} = -0.0592 \text{ in.}$$

$$t_2 = \frac{-12{,}000 \times 3 \times 4 \times 1{,}728}{1{,}050{,}000{,}000} = -0.2368 \text{ in.}$$

$$\Delta = 2t_1 + t_2 = -0.3552 \text{ in.}$$

The deflection at B may also be found by drawing the tangent through either A or B.

A cantilever beam is shown in Fig. 146. In finding t_1,

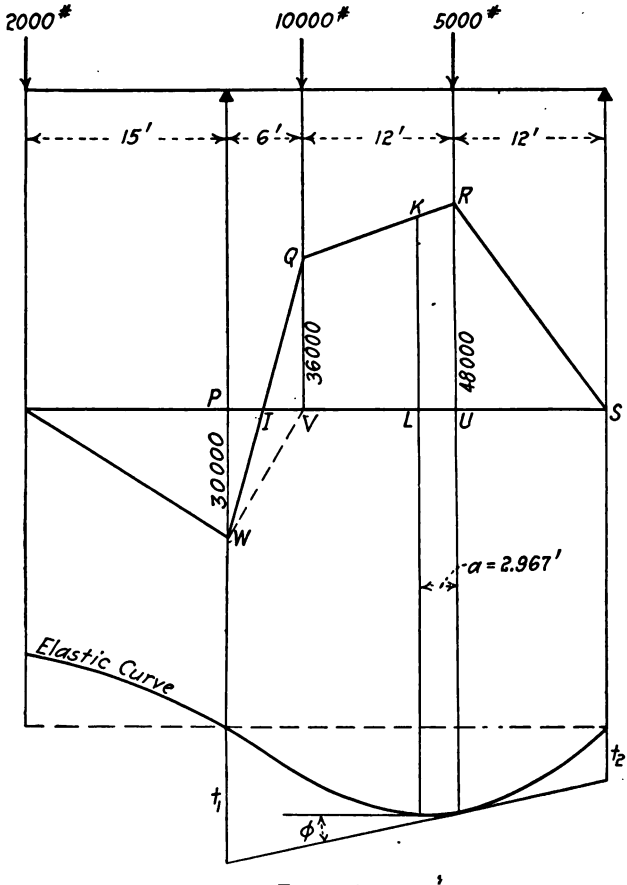


FIG. 146.

positive and negative areas are encountered in the M-diagram. These may be treated in one of two ways. The point of zero bending moment at I may be determined, and the areas PIW and IQV treated separately; or the area WIV may be included with both positive and negative areas as follows:

Area-moment about P.

area $PVW - 30{,}000 \times 3 \times 2 = -180{,}000$
area $WQV + 36{,}000 \times 3 \times 4 = +432{,}000$
area $QVU + 36{,}000 \times 6 \times 10 = +2{,}160{,}000$
area $QRU + 48{,}000 \times 6 \times 14 = +4{,}032{,}000$
$+ 6{,}444{,}000$ ft.^{3}-lb.

If EI is expressed in foot^2-pounds then

$$t_1 = \frac{6{,}444{,}000}{EI} \text{ ft.}$$

Area-moment about S

area SRU $48{,}000 \times 6 \times 8 = 2{,}304{,}000$ ft.^{3}-lb.

$$t_2 = \frac{2{,}304{,}000}{EI} \text{ ft.}$$

$$\phi = \frac{t_1 - t_2}{30} = \frac{138{,}000}{EI}$$

Let KL represent the ordinate in the M-diagram at the point of maximum deflection, then

area $KRUL = 138{,}000$
$a = 2.967$ ft.
$KL = 45{,}033$

The maximum deflection may now be found as in previous cases.

SEC. V. BEAMS WITH VARYING CROSS-SECTION

150. General Expressions.—The moment of inertia of beams having uniform cross-section is constant, and for this reason I appears outside the integral sign in Eqs. (1) and (2). When the cross-section is not uniform the moment of inertia varies, and Eqs. (1) and (2) become

$$\phi = \frac{1}{E}\int_A^B \frac{M\,dx}{I} \tag{8}$$

$$t = \frac{1}{E}\int_A^B \frac{M x\,dx}{I} \tag{9}$$

In order to perform the integration, I as well as M must be expressed as a function of x. This is relatively a simple matter when the beam has a rectangular cross-section varying uniformly in breadth or depth; but this method often results in

long and cumbersome expressions when applied to structural steel sections. In all such instances the geometric method is preferable.

151. Beams with Varying Depth.—The beam in Fig. 147*a* is a plate girder. The ⅜ in. web plate is 24 in. wide at the ends, and the width increases uniformly to 36 in. at the center. Each flange is composed of two angles 5 by 3½ by ⅜ with the

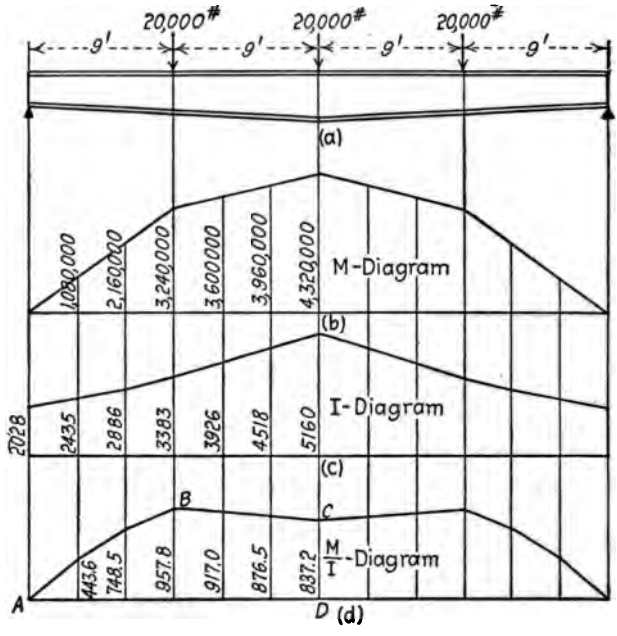


FIG. 147.

3-in. leg against the web. The distance back to back of angles is the width of the web plus ½ in. Ordinates in the M-diagram (Fig. 147*b*) are given in inch-pounds every 3 ft. An I-diagram is shown in Fig. 147*c*. The ordinates represent the moment of inertia in inches[4] at 3-ft. intervals. Each ordinate in the M-diagram has been divided by the corresponding ordinate in the I-diagram, and the quotient recorded in the $\frac{M}{I}$-diagram (Fig. 147*d*). The ordinates in this diagram are expressed in pounds/inches.[3] Since the girder and the loads are symmetrical, the $\frac{M}{I}$-diagram is symmetrical about the vertical ordinate

through the center of the span; the maximum deflection is at the center, and the tangent to the elastic curve (not drawn) at the center is horizontal; hence the tangential deviation t at the left support equals the area-moment $ABCD$ about A divided by E.

Area-moment about A

$$\begin{aligned} 443.6 \times 36 \times \ 36 &= \ 575{,}000 \\ 748.5 \times 36 \times \ 72 &= 1{,}940{,}000 \\ 957.8 \times 36 \times 108 &= 3{,}724{,}000 \\ 917.0 \times 36 \times 144 &= 4{,}754{,}000 \\ 876.5 \times 36 \times 180 &= 5{,}680{,}000 \\ 837.2 \times 18 \times 204 &= 3{,}074{,}000 \\ &\quad 19{,}747{,}000 \text{ lb./in.} \end{aligned}$$

$$\Delta_{max.} = t = \frac{19{,}747{,}000 \text{ lb./in.}}{29{,}000{,}000 \text{ lb./in}^2.} = 0.68 \text{ in.}$$

When the ordinates in the $\frac{M}{I}$-diagram are computed only for the ordinates at B and C, and AB and BC considered as straight lines, the computations are as follows:

Area-moment about A

$$\begin{aligned} 957.8 \times 54 \times \ 72 &= \ 3{,}724{,}000 \\ 957.8 \times 54 \times 144 &= \ 7{,}448{,}000 \\ 837.2 \times 54 \times 180 &= \ 8{,}138{,}000 \\ &\quad 9{,}310{,}000 \end{aligned}$$

$$\Delta_{max.} = \frac{19{,}310{,}000}{29{,}000{,}000} = 0.67 \text{ in.}$$

Thus it is clear that, if the ordinates in the $\frac{M}{I}$-diagram were relatively close together, say at every foot or closer; or even if I were expressed as an exact function of x in Eq. (9) and the integration performed, the results in either case would not differ materially from those obtained above.

152. Beams with Cover Plates.—The plate girder in Fig. 148*a* consists of a 24 by ⅜ web plate, four angles 5 by 3½ by ⅜ and two cover plates 12 by ⅜ by 24 ft. symmetrical about the center line. The M-diagram is shown in Fig. 148*b*, the I-diagram in Fig. 148*c*, and the $\frac{M}{I}$-diagram in Fig. 148*d*.

Area moment about A

$$
\begin{aligned}
1{,}065.1 \times 36 \times 48 &= 1{,}840{,}500 \\
631.6 \times 18 \times 84 &= 955{,}000 \\
947.4 \times 18 \times 96 &= 1{,}637{,}100 \\
947.4 \times 54 \times 144 &= 7{,}367{,}000 \\
1{,}263.2 \times 54 \times 180 &= 12{,}278{,}300 \\
&\underline{\quad\quad} \\
&24{,}077{,}900
\end{aligned}
$$

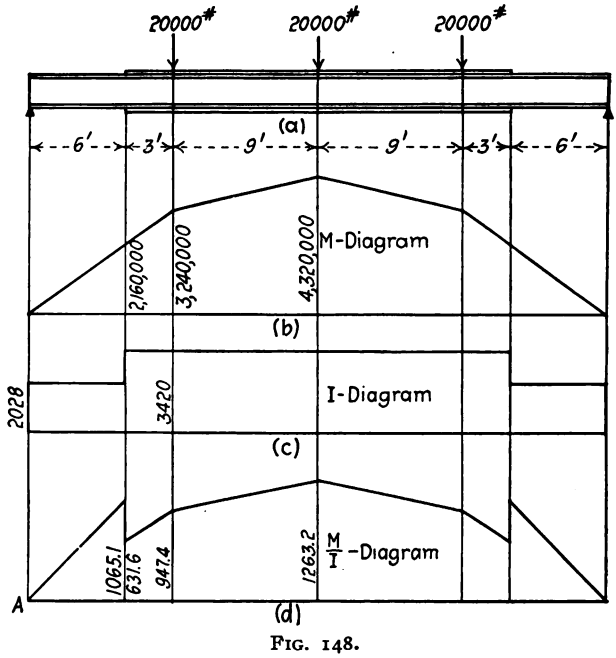


FIG. 148.

The deflection at the center is

$$\Delta = \frac{24{,}077{,}900}{29{,}000{,}000} = 0.83 \text{ in.}$$

The solution by integration may be obtained as follows: Let M_1 represent the bending moment for values of x between 0 and 9; and M_2 the bending moment for values of x between 9 and 18. Then

$$M_1 = 30{,}000x$$
$$M_2 = 10{,}000x + 180{,}000$$

Then for values of x between 0 and 6, $EI_1 = 408{,}417{,}000$ ft^2.-

lb. and for values of x between 6 and 18, $EI_2 = 688{,}750{,}000$ ft^2.-lb. The deflection at the center expressed in feet is

$$\Delta = t = \frac{1}{EI_1}\int_0^6 M_1 x dx + \frac{1}{EI_2}\int_6^9 M_1 x dx + \frac{1}{EI_2}\int_9^{18} M_2 x dx$$

The solution by integration is much more simple in this problem than in the preceding one, for in this problem I is constant between certain limits of x and is therefore not a function of x.

The geometric treatment by area-moments is by far the simplest and best method now known for obtaining a solution of any practical problem involving the deflection of beams.

CHAPTER VI

RESTRAINED AND CONTINUOUS BEAMS

SEC. I. RESTRAINED OR FIXED BEAMS

153. General Considerations.—The beam in Fig. 149 is considered *fixed* or *restrained* at A, if built into the wall in such

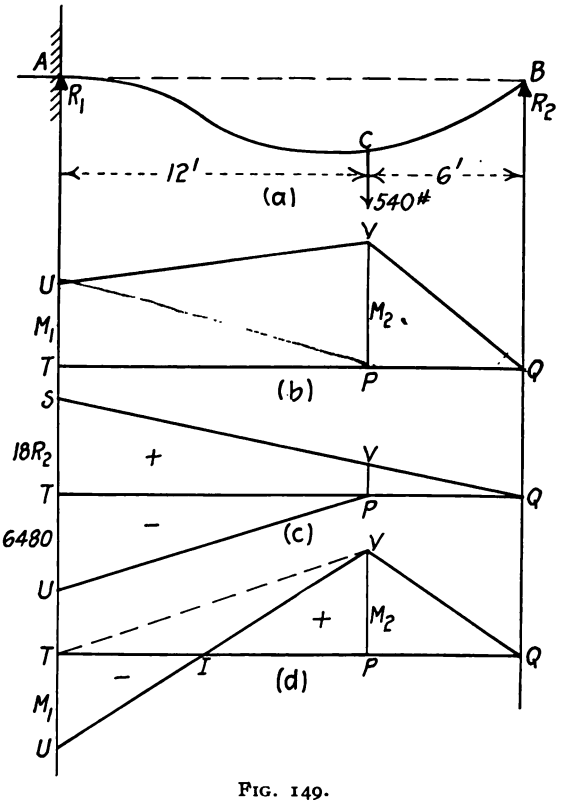


FIG. 149.

a way that any attempt of the external forces to rotate the beam at A is successfully resisted, and the neutral plane of the beam in its original position AB remains tangent to the elastic curve at A when the beam is bent. Under these conditions the reactions R_1 and R_2, and the resisting moment M_1 at A present

more unknown quantities than can be determined by the principles of statics; and the beam is statically indeterminate. The principles of deflections render their most helpful service in the solution of problems of this character. The ease with which problems of this kind are solved, depends to some extent upon the manner in which the M-diagram is drawn; for it may be represented in three ways as shown in Figs. 149*b*, 149*c* or 149*d*; and each case will be considered separately.

154. Restraint at One End—Concentrated Load.—In Fig. 149*b*, let M_1 and M_2 represent the bending moments at A and C respectively. The bending moment at B is zero. The line AB is tangent to the elastic curve at A, and the tangential deviation t at B is zero, therefore

$$t = \frac{1}{EI}\int_A^B Mx\,dx = 0$$

or

$$\int_A^B Mx\,dx = 0$$

Hence the area-moment of $QVUTP$ about Q is zero, or

$$M_2(3 \times 4) + M_2(6 \times 10) + M_1(6 \times 14) = 0$$

$$7M_1 + 6M_2 = 0$$

From statics $M_1 = -(540 \times 12) + 18R_2$

and $M_2 = 6R_2$

whence $-45{,}360 + 126R_2 + 36R_2 = 0$

$$R_2 = 280$$

$$R_1 = 260$$

$$M_1 = -1{,}440$$

and $M_2 = 1{,}680$

In Fig. 149*c*, the M-diagram is drawn in parts; QST is the M-diagram for the reaction at B, and TPU is the M-diagram for the load at C. The area QST is positive and the area TPU is negative. The area-moment of the total diagram about Q is zero; therefore

$$18R_2(9 \times 12) - 6{,}480(6 \times 14) = 0$$

$$R_2 = 280$$

$$TS = 18R_2 = 5{,}040$$

$$PV = \frac{1}{3} \times 5{,}040 = 1{,}680$$

$$SU = 5{,}040 - 6{,}480 = -1{,}440$$

In the two preceding solutions no speculation was made as to the general form of the elastic curve. The curve ACB might have had any shape whatsoever, so long as its tangent at A passes through B. It is not always wise to presume upon the general form of the elastic curve before computations are made; but in the present simple case it is quite safe to assume that the curve is concave on the under side near A, and concave on the upper side at C, with a point of contraflexure between. Hence the bending moment is negative at A, positive at C and zero at an intermediate point I; consequently the M-diagram may be sketched as in Fig. 149*d*. In finding the area-moment, the area TIV, which is not a part of the diagram, can be included as positive area with QVI, and as negative area with TIU.

$$M_2(3 \times 4) + M_2(6 \times 10) - M_1(6 \times 14) = 0$$

$$7M_1 - 6M_2 = 0$$

From statics $$-M_1 = -(540 \times 12) + 18R_2$$

and $$M_2 = 6R_2$$

whence $$R_2 = 280$$

$$R_1 = 260$$

$$-M_1 = -1{,}440$$

$$M_2 = 1{,}680$$

When an unknown ordinate in the M-diagram is represented by a symbol, it is generally better to assume that the ordinate is positive as in Fig. 149*b*; then if the solution shows that the ordinate is negative, the M-diagram may be re-drawn if desirable. Frequently the M-diagram may be constructed to advantage, as shown in Fig. 149*c*.

Only two independent static equations can be written for the solution of a system of parallel forces. In the present problem there were three unknown quantities to be determined, hence one elastic equation was necessary for a solution.

155. Restraint at One End—Uniform Load.—The beam in Fig. 150*a*, fixed at A and simply supported at B, carries a total load W, uniformly distributed. In Fig. 150*b* $\underline{QTU}$ is the M-diagram for the uniform load and QTS is the M-diagram for the reaction at B. If the tangent to the elastic curve is drawn

through A, the tangential deviation t at B is zero, hence the area-moment of QSU about Q is zero.

$$(R_2 l)\left(\tfrac{1}{2}l\right)\left(\tfrac{2}{3}l\right) - \left(\frac{Wl}{2}\right)\left(\frac{l}{3}\right)\left(\tfrac{3}{4}l\right) = 0$$

$$R_2 = \tfrac{3}{8}W$$

$$R_1 = \tfrac{5}{8}W$$

$$TS = \tfrac{3}{8}Wl$$

The M-diagram may now be drawn to scale as in Fig. 150c;

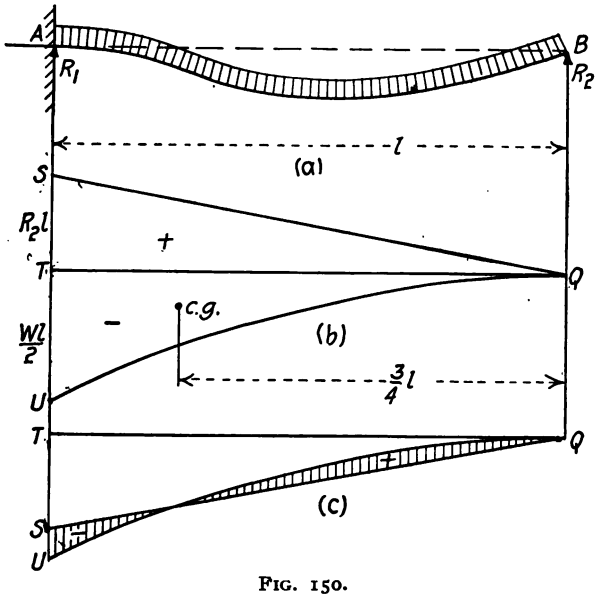


FIG. 150.

the resisting moment at A is

$$SU = TS - TU = \tfrac{3}{8}Wl - \tfrac{1}{2}Wl = -\tfrac{1}{8}Wl$$

156. Restraint at Both Ends—Concentrated Load.—The beam in Fig. 151*a* is fixed at each end. The M-diagram is sketched in Fig. 151*b* by assuming all ordinates positive. It cannot be drawn to scale until M_1, M_2 and M_3 are known. There are four unknown quantities involved in the external

forces acting at the points of support, R_1, R_2, M_1 and M_2 for which four independent equations are necessary. The two static equations may be written thus,

$$R_1 + R_2 = P \qquad (1)$$

$$M_3 = klR_1 + M_1 = (1 - k)lR_2 + M_2 \qquad (2)$$

Two elastic equations are necessary. Let ϕ be the angle

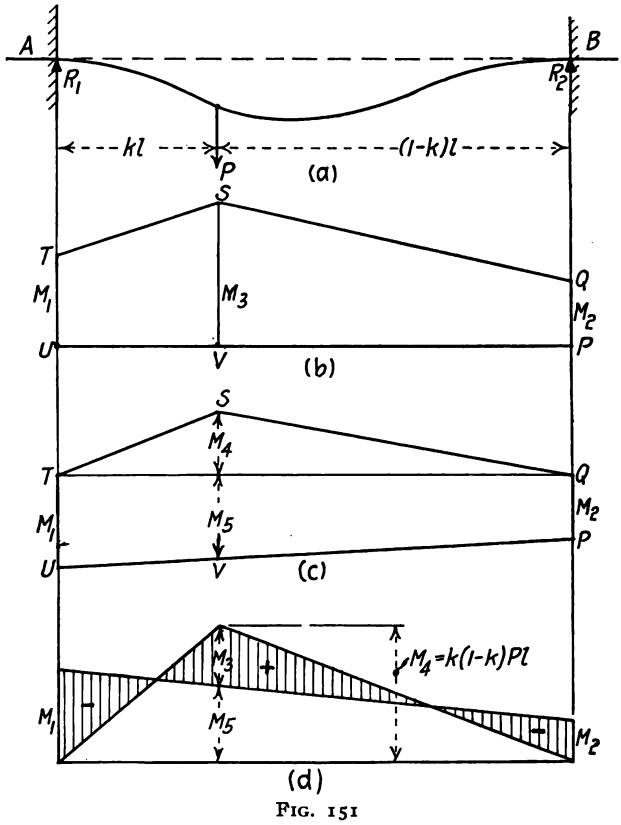


FIG. 151

which the tangent through A makes with the tangent through B; then, since $\phi = 0$, the area of the M-diagram between A and B is zero; therefore

$$\frac{M_1 + M_3}{2} kl + \frac{M_3 + M_2}{2} (1 - k)l = 0 \qquad (3)$$

The line AB is tangent to the elastic curve at B, and the

tangential deviation at A is $t_1 = 0$; therefore the area-moment of the M-diagram about TU is zero, hence

$$\left(\frac{M_1kl}{2}\right)\left(\frac{kl}{3}\right)+\left(\frac{M_3kl}{2}\right)\left(\frac{2kl}{3}\right)+\frac{M_3(1-k)l}{2}\left[kl+\frac{1}{3}(1-k)l\right]$$
$$+\frac{M_2(1-k)l}{2}\left[kl+\frac{2}{3}(1-k)l\right]=0 \qquad (4)$$

A third elastic equation may be written by equating to zero the area-moment of the M-diagram about QP; but obviously this equation would not be independent of Eqs. (3) and (4). Eqs. (3) and (4) may be reduced to

$$kM_1+(1-k)M_2+M_3=0 \qquad (3a)$$
$$k^2M_1+(2-k-k^2)M_2+(1+k)M_3=0 \qquad (4a)$$

Solving Eqs. (1), (2), (3a) and (4a)

$$R_1=(1-3k^2+2k^3)P$$
$$R_2=(3k^2-2k^3)P$$
$$M_1=-k(1-k)^2Pl$$
$$M_2=-k^2(1-k)Pl$$
$$M_3=2k^2(1-k)^2Pl$$

Since the limits of k are 0 and 1, it is clear that M_1 and M_2 are negative bending moments and M_3 is a positive bending moment. The M-diagram in Fig. 151c is the same as in Fig. 151b, except that TQ has been drawn in the horizontal position. Let

$$SV=M_3=M_4+M_5$$

From geometry

$$M_5=M_2+(1-k)(M_1-M_2)$$

hence

$$M_4=M_3-M_5=k(1-k)Pl$$

If the beam were simply supported, not fixed, at A and B, the bending moment at C would be $M_4=k(1-k)Pl$, therefore TSQ is the M-diagram when the beam is not restrained at the ends by M_1 and M_2. In a numerical problem the M-diagram should be sketched as in Fig. 151c, and M_4 computed as for a simple beam. After the negative moments M_1 and M_2 have been determined, the trapezoid $TQPU$ may be *revolved* about TQ and the diagram drawn to scale as shown in Fig. 151d.

The reactions and resisting moments will be determined for the fixed beam in Fig. 152a.

From statics $$SV = \frac{640}{24} \times 9 \times 15 = 3{,}600$$

$$M_1 = -(640 \times 9) + 24R_2 + M_2$$

or $$R_2 = \frac{5{,}760 + M_1 - M_2}{24}$$

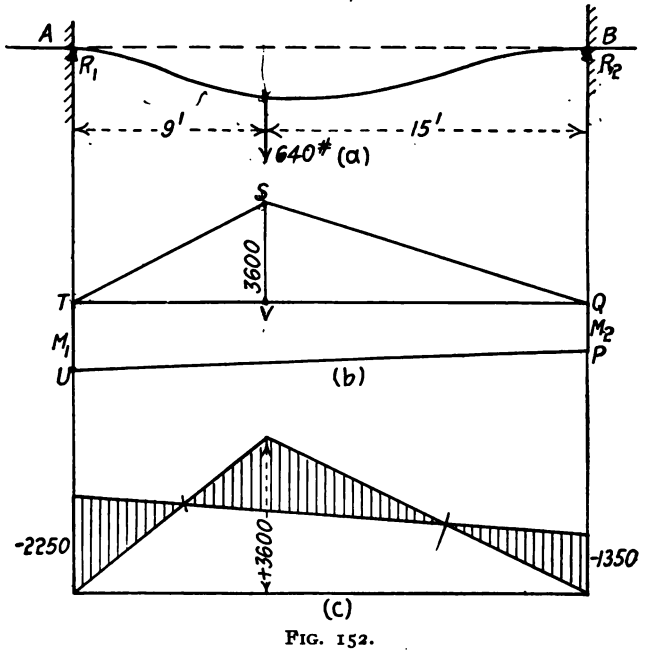


FIG. 152.

The angle between the tangents through A and B is

$$\phi = 0$$

or $$\frac{(M_1 + M_2)24}{2} + \frac{3{,}600 \times 24}{2} = 0$$

The tangential deviation at B is

$$t_2 = 0$$

or $$\frac{3{,}600 \times 15 \times 10}{2} + \frac{3{,}600 \times 9 \times 18}{2} + \frac{24M_2 \times 8}{2} + \frac{24M_1 \times 16}{2} = 0$$

whence

$$M_1 = -2{,}250$$
$$M_2 = -1{,}350$$
$$R_1 = 437.5$$
$$R_2 = 202.5$$

These results may be checked by the formulas of the preceding problem. The M-diagram is drawn to scale in Fig. 152*c*.

157. Restraint at Both Ends—Uniform Load.—The fixed beam in Fig. 153 supports a total load W, uniformly distributed. Since the loading is symmetrical, the resisting moment and reactions at B are the same as at A. The area TSQ is the

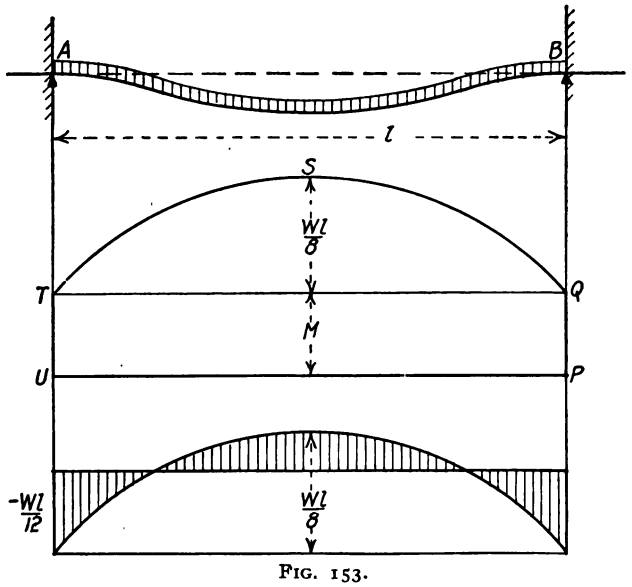


FIG. 153.

M-diagram for a simply supported beam and the area $TUPQ$ represents the resisting moment M at each end. The angle ϕ between the tangents through A and B is zero, consequently the area of the M-diagram is zero, therefore

$$Ml + \left(\frac{Wl}{8}\right)\left(\frac{2}{3}l\right) = 0$$

$$M = -\frac{Wl}{12}$$

The bending moment at the center is

$$\frac{Wl}{8} - \frac{Wl}{12} = \frac{Wl}{24}$$

and the M-diagram may be drawn to scale as shown.

SEC. II. CONTINUOUS BEAMS

158. Continuous beams rest on more than two supports and have more than one span. Consequently they are statically indeterminate, and one or more elastic equations are necessary for finding the reactions. An introduction to the

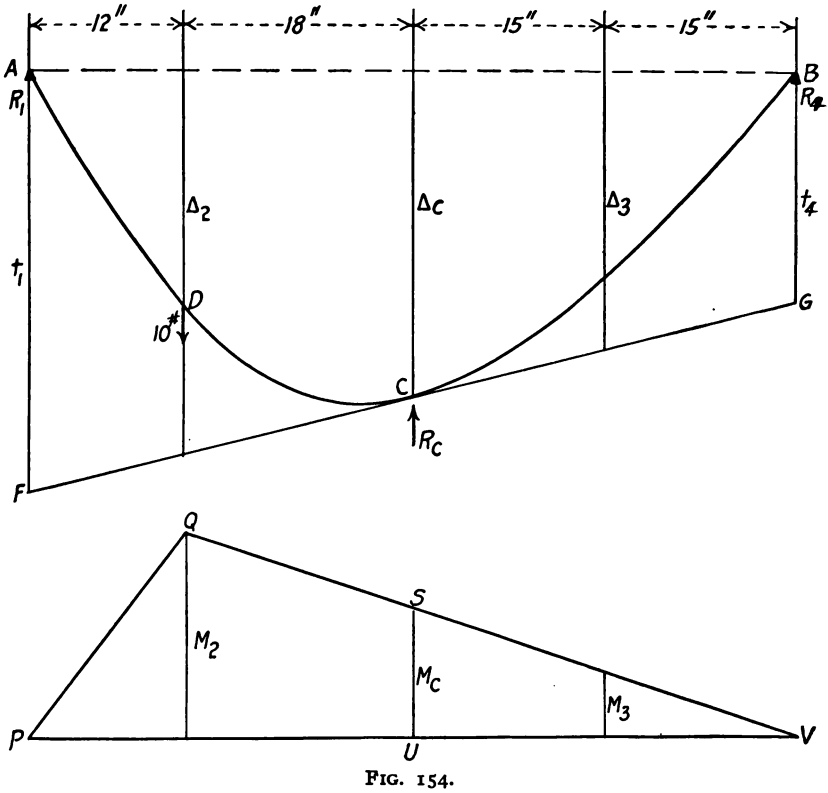


FIG. 154.

problem will be made in connection with the beam in Fig. 154, which supports a load of 10 lb. at D. The tangent FG is drawn to the elastic curve through C at the middle of the span. The product EI of the modulus of elasticity, and the moment of inertia is assumed as unity, and the weight of the beam is not considered. The M-diagram is PQV. When the beam is supported at A and B only, the reactions R, the bending moments M, the tangential deviations t, and the deflections Δ are as tabulated in the column 1 of Table I. Now suppose that an

upward force $R_c = 1$ lb. is applied at C. The shape of the elastic curve will be altered in accordance with the data given in column 2. The moments are statically determinate; and if a new M-diagram is drawn to scale an angle will appear at S,

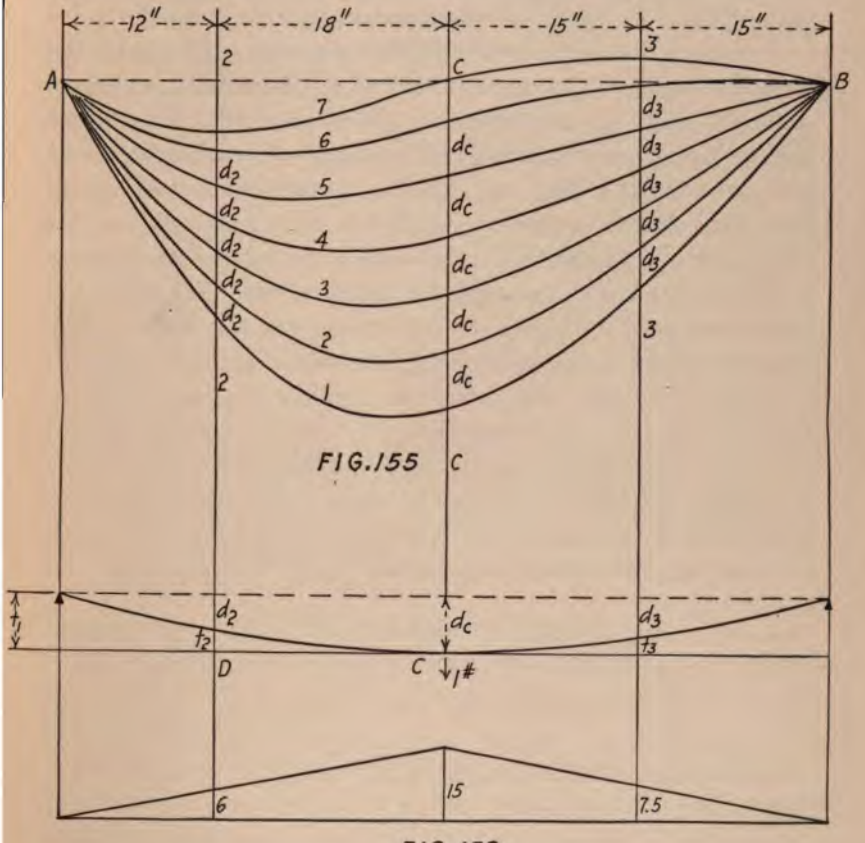


FIG. 156

the line QV no longer remaining straight. The quantities t and Δ (column 2) are computed from this diagram. Columns 3, 4, 5 and 6 give similar data, as the upward force at C is increased by 1-lb. increments. The table clearly shows that for each increase of 1 lb. in R_c, Δ_c is decreased 4,500 units. The original deflection Δ_c was (from column 1) 25,560 units; and it is clear that if the point C is raised 4,500 units by each in-

crease of 1 lb. in R_c, the force necessary to raise the point C — to a level with A and B is

$$R_c = \frac{25{,}560}{4{,}500} = 5.68 \text{ lb.}$$

By balancing the moments about either A or B, the end reactions, R_1 and R_4, may be determined as given in column 7.

A very interesting study of the successive stages in the transformation of the elastic curve may be made, if diagrams similar to Fig. 154 are drawn to scale in accordance with the data given in each column of the table. These curves have been combined in Fig. 155, and are numbered to correspond with the columns in the table. When $R_c = 4$ lb., $R_1 = 6$ lb., and $R_4 = 0$; consequently $M_c = M_3 = 0$ and the M-diagram is PQS (Fig. 154), the point S coinciding with U. Since the area-moment of VSU about V is zero, $t_4 = 0$, G coincides with B; and the elastic curve CB being bent by no moment becomes a straight line coinciding with the tangent CG; (see curve 5). When $R_c = 5$ lb.; R_4, being negative, acts downward; M_c and M_3 become negative moments; and t_4, being negative for curve 6, is measured above B instead of below. Finally, when $R_c = 5.68$ lb., and the point C has been raised to the line AB (curve 7); the tangent through C makes equal intercepts on the

TABLE I

	1	2	3	4	5	6	7
	$R_c = 0$	$R_c = 1$	$R_c = 2$	$R_c = 3$	$R_c = 4$	$R_c = 5$	$R_c = 5.68$
R_1	8	7.5	7.0	6.5	6.0	5.5	5.16
R_4	2	1.5	1.0	0.5	0.0	−5.0	−0.84
M_2	96	90	84	78	72	66	61.92
M_c	60	45	30	15	0	−15	−25.2
M_3	30	22.5	15	7.5	0	−7.5	−12.6
t_1	33,120	28,620	24,120	19,620	15,120	10,620	7,560
t_2	11,664	9,720	7,776	5,832	3,888	1,944	622.08
t_3	5,625	4,218.75	2,812.5	1,406.25	0	−1,406.25	−2,362.5
t_4	18,000	13,500	9,000	4,500	0	−4,500	−7,560.
Δ_2	18,432	15,876	13,320	10,764	8,208	5,652	3,913.93
Δ_c	25,560	21,060	16,560	12,060	7,560	3,060	0
Δ_3	16,155	13,061.25	9,967.5	6,873.75	3,780	686.25	−1,417.5

ordinates through A and B, the one below and the other above AB, hence $t_1 = -t_4$.

Let d_2, d_c and d_3 represent respectively the differences in Δ_2, Δ_c and Δ_3, for consecutive columns 1 to 6 in Table I; then $d_2 = 2{,}556$; $d_c = 4{,}500$ and $d_3 = 3{,}093.75$. These differences in deflection between any two curves 1 to 6 (Fig. 155) are caused by a difference of 1 lb. in the force R_c, irrespective of the magnitudes R_1, R_c or R_4. The fact that these differences are constant for any ordinate whether 2-2, C-C, 3-3 or any other ordinate between A and B which might be chosen, is due to the constant differences in M_2, M_c and M_3. Let m_2, m_c and m_3 represent these differences; then from Table I, $m_2 = 6$ $m_c = 15$ and $m_3 = 7.5$. By reference to Fig. 156 it is at once, apparent that m_2, m_c and m_3 represent the corresponding ordinates in the M-diagram for the beam in question; when supporting a single load of 1 lb. at the center. If EI is again taken as unity, then

$$
\begin{aligned}
d_c = t_1 &= 15 \times 15 \times 20 = 4{,}500 \\
t_2 &= (6 \times 9 \times 6) + (15 \times 9 \times 12) = 1{,}944 \\
t_3 &= (7.5 \times 7.5 \times 5) + (15 \times 7.5 \times 10) = 1{,}406.25 \\
d_2 &= t_1 - t_2 = 2{,}556 \\
d_3 &= t_1 - t_3 = 3{,}093.75
\end{aligned}
$$

Consequently the difference d between any two curves from 1 to 6 (Fig. 155) on any ordinate, caused by a difference of 1 lb. in R_c, equals the deflection at the same ordinate caused by a load of 1 lb. at C (Fig. 156).

Hence, if a beam is continuous over three supports, the intermediate reaction R_c at C (not necessarily at the center) may be determined as follows: Remove the intermediate reaction; find the end reactions as for a simple beam and compute the deflection Δ at C; finally, compute the deflection d at C due to a load of 1 lb. at C. Then

$$R_c = \frac{\Delta}{d}$$

159. Application of Maxwell's Theorem.—Maxwell's theorem of reciprocal displacements may be used in finding the reactions of a continuous beam on three supports. The deflection at D

(Fig. 156) caused by a load of 1 lb. at C is $d_2 = 2{,}556$; hence, the deflection at C caused by a load of 1 lb. at D is 2,556; and the deflection at C caused by 10 lb. at D is $\Delta_c = 2{,}556 \times 10 = 25{,}560$, as shown in Fig. 154. Since the deflection at C, caused by a force of 1 lb. at C, is $d_c = 4{,}500$; then the force at C necessary to raise the point C to a level with AB is

$$R_c = \frac{10d_2}{d_c} = \frac{25{,}560}{4{,}500} = 5.68 \text{ lb.}$$

Hence, if a beam is continuous over three supports at A,C and B, and supports loads P_1, P_2 and P_3 at any points 1, 2 and 3, the intermediate reaction R_c may be found as follows: Remove the loads P and the reaction R_c; place 1 lb. at C, and compute the deflections Δ_1, Δ_2, Δ_3 and Δ_c at the points 1, 2, 3 and C. Then

$$R_c = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3}{\Delta_c}$$

The reactions at A and B may be determined by statics after the reaction at C is known. It was not necessary to apply Maxwell's theorem to the intermediate reaction, for either end reaction might have been determined equally as well by this theorem.

The ratio of d_2 to d_c (Fig. 156) may be determined from Table I, page 222. The load 1 lb. is at the center, hence $k = 0.5$. When $c = 0.2$, $F = 0.071$; when $c = 0.5$, $F = 0.125$; hence

$$\frac{d_2}{d_c} = \frac{0.071}{0.125}$$

and

$$R_c = \frac{10d_2}{d_c} = 5.68 \text{ lb.}$$

160. The Conventional Method.—The method which is generally employed in finding the reactions, by area-moments, of a beam continuous over three supports, will be given in connection with Fig. 157. The elastic curve is not shown and no speculation with reference to its form will be made. Let FG represent the tangent to the elastic curve drawn through C. Its slope is unknown; it may be positive or negative. One thing is certain. Since the two spans are equal in length, the

tangential deviations t_1 and t_2 are equal in magnitude. They have opposite signs, since one is measured above the line AB

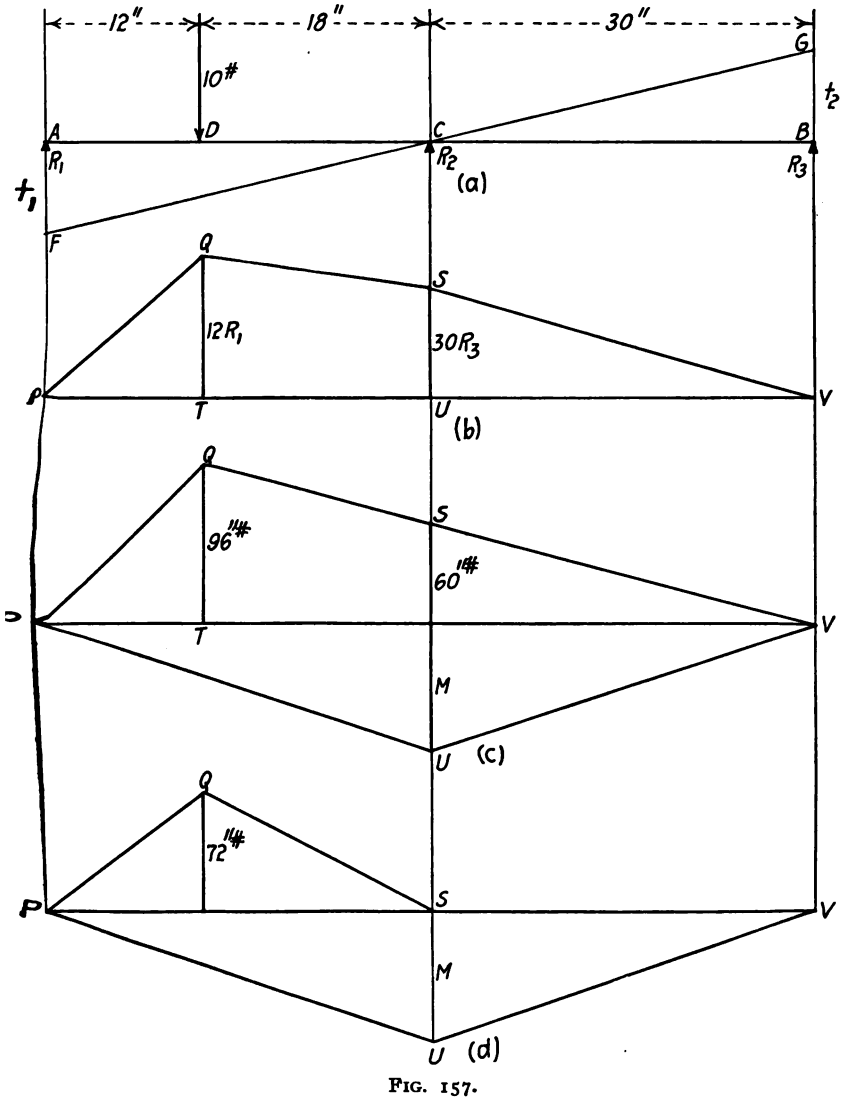


FIG. 157.

and the other below it.

Hence

$$t_1 = -t_2$$

All reactions will be assumed to act upward, hence a negative numerical value for the solution of a reaction indicates that its action is downward.

The bending moment is known only at two points A and B. The M-diagram may be constructed in several ways.

In Fig. 157*b*, all ordinates are expressed in terms of the end reactions.

$$EIt_1 = 12R_1(6 \times 8) + 12R_1(9 \times 18) + 30R_3(9 \times 24)$$
$$EIt_2 = 30R_3(15 \times 20)$$
$$t_1 = -t_2$$

whence $7R_1 = -43R_3$

From statics $30R_1 - 180 = 30R_3$

$$R_1 = R_3 + 6$$

Therefore
$$R_1 = 5.16$$
$$R_2 = 5.68$$
$$R_3 = -0.84$$

In Fig. 157*c*, the M-diagrams for the load at D and for the reaction at C are sketched separately. PQV is the M-diagram when the center reaction is removed and AB considered as a simple beam, supporting the load at D. PUV is the M-diagram when the load at D is removed, and AB considered as a simple beam held in equilibrium by forces at A, B and C.

$$EIt_1 = (96 \times 6 \times 8) + (96 \times 9 \times 18) + (60 \times 9 \times 24) + M(15 \times 20)$$
$$EIt_2 = (60 \times 15 \times 20) + M(15 \times 20)$$
$$t_1 = -t_2$$

whence
$$M = -85.2$$

The bending moment at C is

$$M = SU = 60 - M = -25.2$$

hence $-25.2 = 30R_1 - 180 = 30R_3$

Therefore
$$R_1 = 5.16$$
$$R_2 = 5.68$$
$$R_3 = -0.84$$

In Fig. 157*d*, the M-diagram is drawn by first considering that AC and CB are simple beams; *i.e.*, by assuming no continuity and no bending moment at C; thus PQS is the M-

diagram for the simple beam AC supporting 10 lb. at D. There is no corresponding diagram for CB, since there is no load in that span. The area PUV is then added to provide for the bending moment on account of continuity at C.

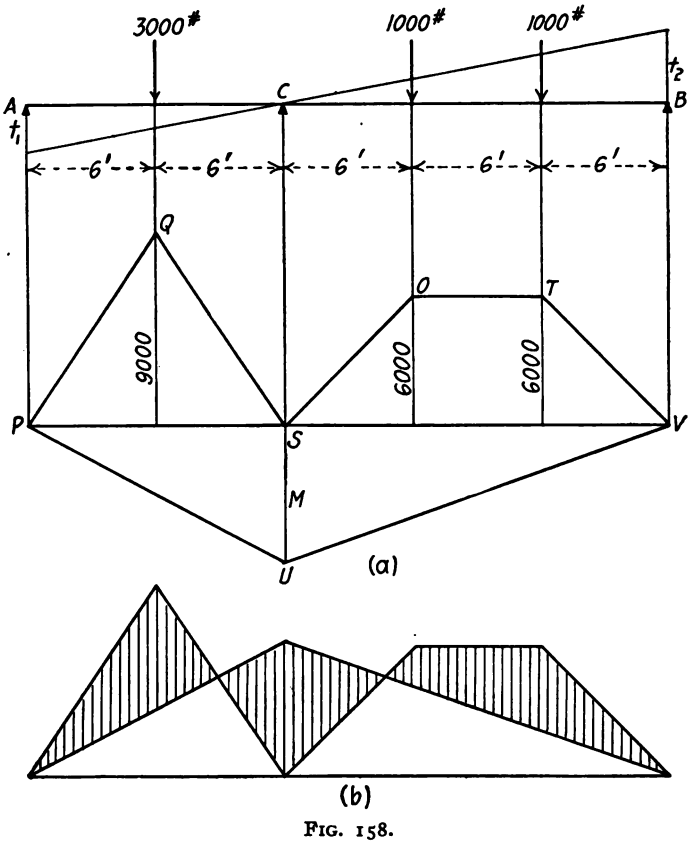


FIG. 158.

$$EIt_1 = (72 \times 6 \times 8) + (72 \times 9 \times 18) + M(15 \times 20)$$
$$EIt_2 = M(15 \times 20)$$
$$t_1 = -t_2$$

whence

$M = -25.2$ as before.

After the reactions have been determined, the bending moments at C and D, and the deflection at any point may be computed. The elastic curve when drawn will have the general configuration shown in Fig. 159.

161. In Fig. 158*a*, *PQS* is the M-diagram when *AC* is considered as a simple beam, and *SOTV* is the M-diagram when *CB* is considered as a simple beam; then the area *PUV* is added to provide the bending moment on account of the continuity. Let the tangent to the elastic curve be drawn through *C* and let t_1 and t_2 represent the tangential deviations at *A* and *B*

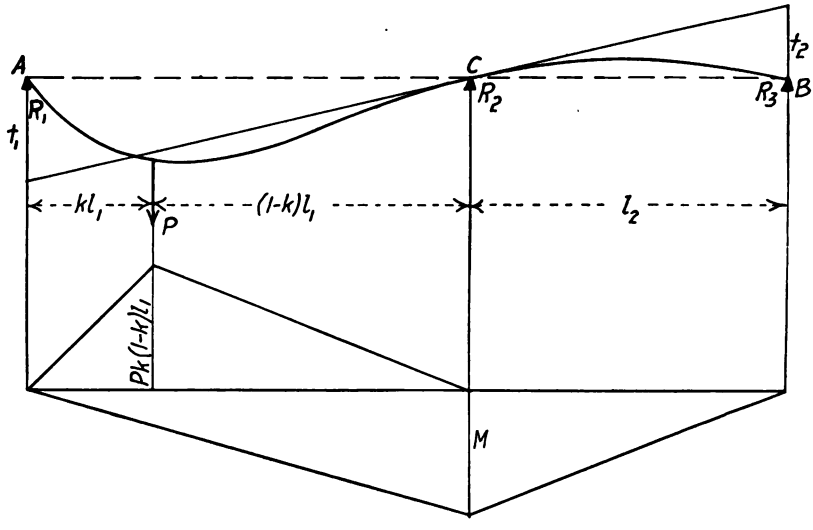


FIG. 159.

respectively, then

$$EIt_1 = (9{,}000 \times 6 \times 6) + M(6 \times 8)$$

$$EIt_2 = (6{,}000 \times 3 \times 4) + (6{,}000 \times 6 \times 9) + (6{,}000 \times 3 \times 14) + M(9 \times 12)$$

$$t_1 : -t_2 :: 12 : 18$$

whence

$$M = -6{,}300$$

The M-diagram may now be drawn to scale as shown in Fig. 158*b*.

From statics

$$-6{,}300 = 12R_1 - 18{,}000$$

$$R_1 = 975$$

$$-6{,}300 = 18R_3 - 6{,}000 - 12{,}000$$

$$R_3 = 650$$

$$R_2 = 5{,}000 - 975 - 650 = 3{,}375$$

162. The general expressions for R_1, R_2 and R_3 will now be developed in connection with Fig. 159. Let the tangent to the elastic curve be drawn through C and let t_1 and t_2 represent the tangential deviations at A and B respectively; then

$$EIt_1 = Pk(1-k)l_1\left(\tfrac{1}{2}l_1\right)\tfrac{2}{3}\left[kl_1 + \tfrac{1}{2}(1-k)l_1\right] + M\left(\tfrac{1}{2}l_1\right)\left(\tfrac{2}{3}l_1\right)$$

$$= \frac{Pl^3}{6}(k - k^3) + \frac{Ml_1^2}{3}$$

$$EIt_2 = M\left(\tfrac{1}{2}l_2\right)\left(\tfrac{2}{3}l_2\right) = \frac{Ml_2^2}{3}$$

$$t_1 : -t_2 :: l_1 : l_2$$

$$t_1 l_2 = -t_2 l_1$$

$$M = -\frac{Pl_1^2(k - k^3)}{2(l_1 + l_2)} \tag{5}$$

$$R_1 = P(1-k) - \frac{Pl_1(k - k^3)}{2(l_1 + l_2)} \tag{6}$$

$$R_2 = Pk + \frac{Pl_1(k - k^3)}{2l_2} \tag{7}$$

$$R_3 = -\frac{Pl_1^2(k - k^3)}{2(l_1 + l_2)l_2} \tag{8}$$

Since k is less than unity, $k - k^3$ is positive; hence M is a negative bending moment, and R_3 is a negative reaction acting downward.

When the two spans are of equal length l, the reactions are

$$R_1 = \frac{P}{4}(k^3 - 5k + 4) \tag{9}$$

$$R_2 = \frac{P}{4}(-2k^3 + 6k) \tag{10}$$

$$R_3 = \frac{P}{4}(k^3 - k) \tag{11}$$

163. In Fig. 160*a* the parabola PQS is the M-diagram, when AC is considered as a simple beam; and the parabola STV is the M-diagram when CB is considered as a simple beam. The triangle PUV is added to represent the bending moment on account of the continuity. If the tangent to the elastic curve

be drawn through C, and t_1 and t_2 represent the tangential deviations at A and B respectively; then

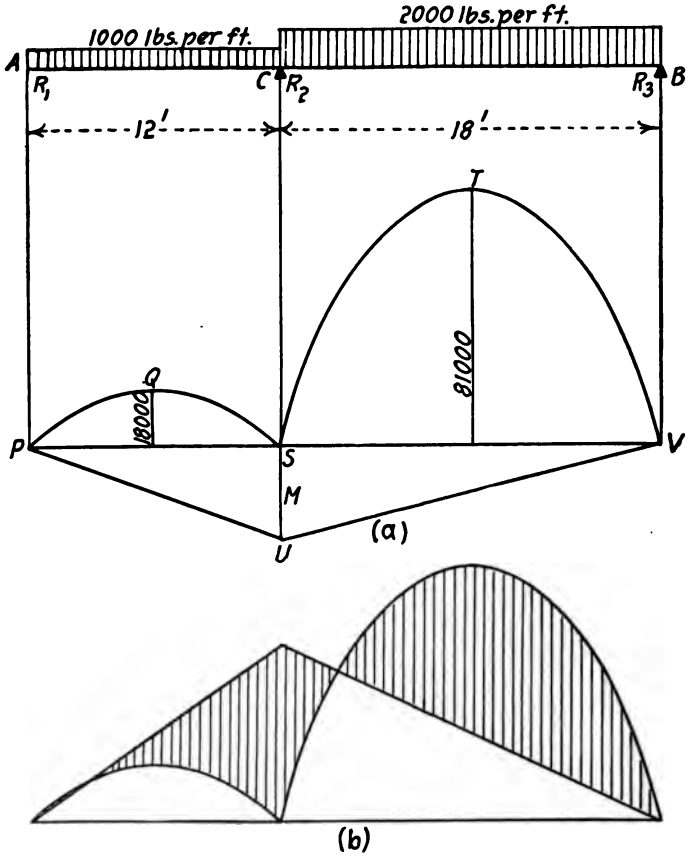


FIG. 160.

$$EIt_1 = (18{,}000 \times 12 \times \tfrac{2}{3} \times 6) + M(6 \times 8)$$
$$= 864{,}000 + 48M$$
$$EIt_2 = (81{,}000 \times 18 \times \tfrac{2}{3} \times 9) + M(9 \times 12)$$
$$= 8{,}748{,}000 + 108M$$
$$3t_1 = -2t_2$$
$$M = -55{,}800$$

The M-diagram may now be drawn to scale as shown in Fig. 160*b*.

From statics $-55{,}800 = 12R_1 - (12{,}000 \times 6) = 18R_3 - (36{,}000 \times 9)$

$$R_1 = 1{,}350$$
$$R_2 = 31{,}750$$
$$R_3 = 14{,}900$$

164. Two Unequal Spans, Supporting Unequal Uniform Loads.—The general expressions for R_1, R_2 and R_3 for a continuous beam of two unequal spans l_1 and l_2, supporting unequal uniform loads, w_1 and w_2 per unit of length, will now be developed in connection with Fig. 161. The M-diagram is drawn as in the preceding problem. If the tangent to the elastic curve is drawn through C; and t_1 and t_2 represent the tangential deviations at A and B respectively; then

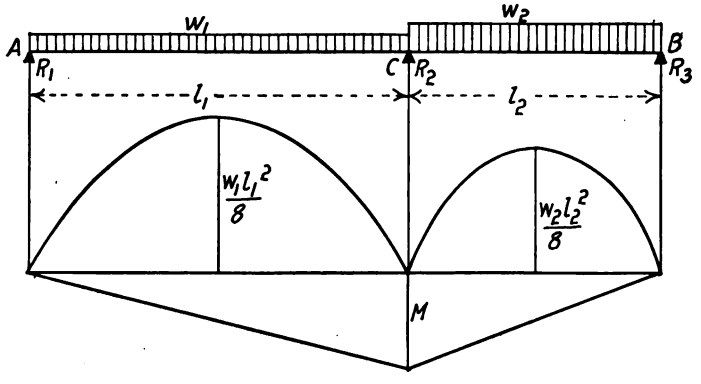


FIG. 161.

$$EIt_1 = \left(\frac{w_1 l_1^2}{8}\right)\left(\frac{2}{3}l_1\right)\left(\frac{1}{2}l_1\right) + \left(\frac{Ml_1}{2}\right)\left(\frac{2}{3}l_1\right)$$

$$= \frac{w_1 l_1^4}{24} + \frac{Ml_1^2}{3}$$

$$EIt_2 = \frac{w_2 l_2^2}{24} + \frac{Ml_2^2}{3}$$

$$t_1 l_2 = -t_2 l_1$$

$$M = -\frac{w_1 l_1^3 + w_2 l_2^3}{8(l_1 + l_2)} \qquad (12)$$

When the spans are equal in length l and the uniform load w per unit of length is the same in both spans, Eq. (12) reduces to

$$M = -\frac{wl^2}{8} \tag{13}$$

165. When a continuous beam supports a combination of uniform and concentrated loads, it will be found expedient to sketch the M-diagram in parts as shown in Fig. 162. The portion (*a*) is the M-diagram for the concentrated loads when no continuity is considered at C; and the portion (*b*) is a similar

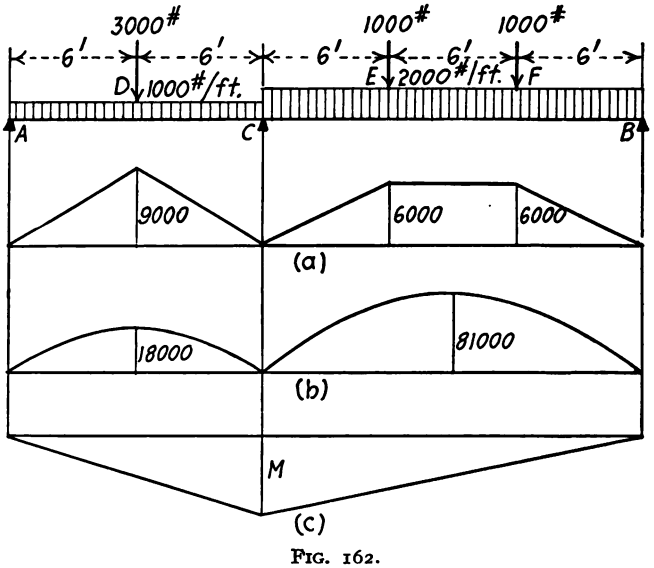


FIG. 162.

diagram for the uniform loads. The continuity is provided for by the portion (*c*). If the tangent to the elastic curve is drawn through C; and t_1 and t_2 represent the tangential deviations at A and B; then

$$EIt_1 = (9{,}000 \times 6 \times 6) + (18{,}000 \times 12 \times \tfrac{2}{3} \times 6) +$$
$$M(6 \times 8) = 1{,}188{,}000 + 48M$$
$$EIt_2 = (6{,}000 \times 3 \times 4) + (6{,}000 \times 6 \times 9) +$$
$$(6{,}000 \times 3 \times 14) + \left(81{,}000 \times 18 \times \tfrac{2}{3} \times 9\right) +$$
$$M(9 \times 12) = 9{,}396{,}000 + 108M$$
$$3t_1 = -2t_2$$
$$M = -62{,}100$$

The reactions may now be determined by the principles of statics.

The value of M may also be determined from Eqs. (5) and (12). Eq. (5) is applicable to the concentrated loads. For the load at D; $P = 3{,}000$, $k = \frac{1}{2}$, $l_1 = 12$ and $l_2 = 18$; hence

$$M = -\frac{3{,}000 \times 144\left(\frac{1}{2} - \frac{1}{8}\right)}{2(12 + 18)} = -2{,}700$$

For the load at F; $P = 1{,}000$, $k = \frac{1}{3}$, $l_1 = 18$ and $l_2 = 12$; hence

$$M = -\frac{1{,}000 \times 324\left(\frac{1}{3} - \frac{1}{27}\right)}{2(12 + 18)} = -1{,}600$$

For the load at E; $P = 1{,}000$, $k = \frac{2}{3}$, $l_1 = 18$ and $l_2 = 12$, hence

$$M = -\frac{1{,}000 \times 324\left(\frac{2}{3} - \frac{8}{27}\right)}{2(12 + 18)} = -2{,}000$$

Hence the bending moment at C, due to the three concentrated loads,

$$M = -2{,}700 - 1{,}600 - 2{,}000 = -6{,}300$$

which agrees with bending moment at C for the beam in Fig. 158.

Equation (12) is applicable to the uniform loads, where $w_1 = 1{,}000$, $w_2 = 2{,}000$, $l_1 = 12$ and $l_2 = 18$; hence

$$M = -\frac{(1{,}000 \times 12^3) + (2{,}000 \times 18^3)}{8(12 + 18)} = -55{,}800$$

which agrees with the bending moment at C for the beam in Fig. 160. The total bending moment at C for the combined uniform and concentrated loads is

$$M = -6{,}300 - 55{,}800 = -62{,}100$$

as previously determined.

166. The continuous beam in Fig. 163 supports a uniform load of 1,000 lb. per foot over a part of the span AC. The area $PQSTW$ is the M-diagram, when AC is considered as a simple

span. Let the tangent to the elastic curve be drawn through C, and let t_1 and t_2 represent the tangential deviations at A and B respectively. In finding the area-moment of $PQSTW$ about P, the parabolic area QST is encountered. This area has all the properties of the area $Q'S'T'$, which is the M-diagram for a simple beam 12 ft. long supporting a uniform load of 1,000

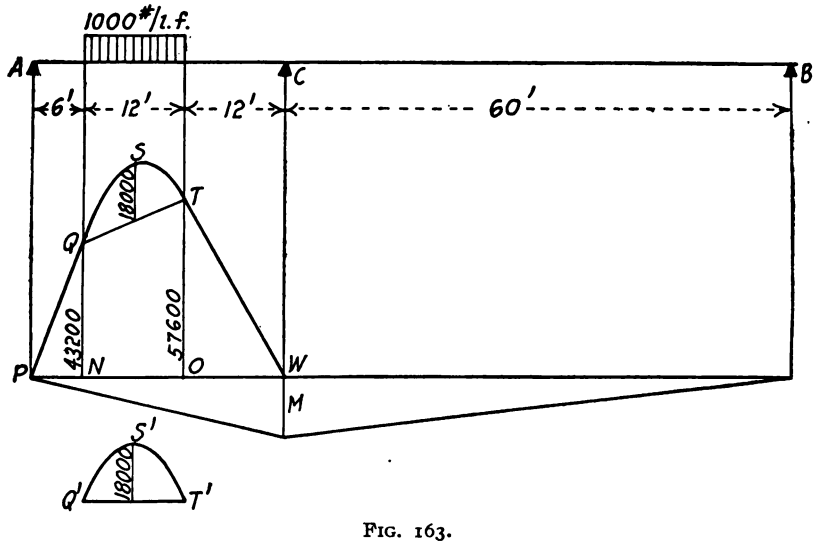


FIG. 163.

lb. per foot over its entire length; hence the area-moment of $PQSTW$ about P may be found as follows:

area PQN	$43{,}200 \times 3 \times 4$	$= 518{,}400$
area QNO	$43{,}200 \times 6 \times 10$	$= 2{,}592{,}000$
area QTO	$57{,}600 \times 6 \times 14$	$= 4{,}838{,}400$
area TOW	$57{,}600 \times 6 \times 22$	$= 7{,}603{,}200$
area QST	$18{,}000 \times 12 \times \frac{2}{3} \times 12$	$= 1{,}728{,}000$
		$17{,}280{,}000$

$$EIt_1 = 17{,}280{,}000 + M(15 \times 20)$$
$$EIt_2 = M(30 \times 40)$$
$$2t_1 = -t_2$$
$$M = -19{,}200$$

The reactions are statically determinate when M is known. The value of M may also be determined by the use of Eq. (5)

in which P is a concentrated load at the distance kl_1 from A. In the present case let P represent the weight of an element, of length dkl_1 at the distance kl_1 from A, then

$$P = 1{,}000\ l_1 dk$$

whence

$$dM = -\frac{1{,}000 l_1^3 (k - k^3) dk}{2(l_1 + l_2)}$$

The value of M may be found by integrating between the limits $k = 0.2$ and $k = 0.6$, hence

$$M = -\int_{0.2}^{0.6} \frac{1{,}000 \times 30^3}{2(30 + 60)} (k - k^3) dk$$

$$= -150{,}000 \left[\frac{k^2}{2} - \frac{k^4}{4}\right]_{0.2}^{0.6}$$

$$= -19{,}200$$

167. The beam in Fig. 164 is continuous over four supports. Two elastic equations are required, in addition to the two

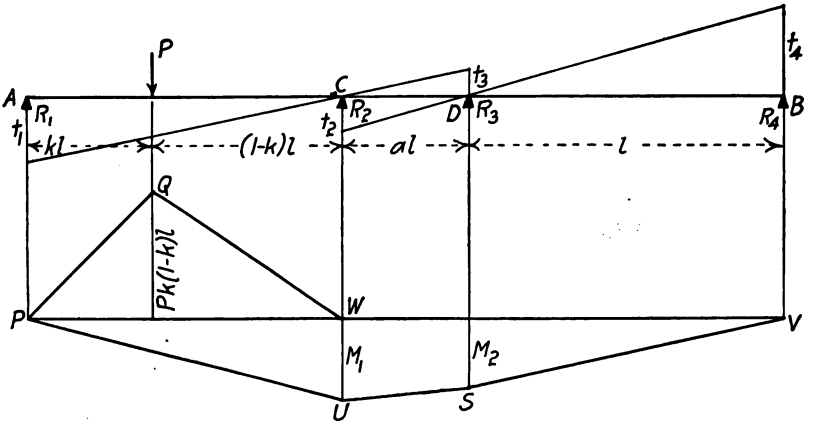


FIG. 164.

static equations which may be written, for the determination of R_1, R_2, R_3 and R_4. Let t_1 and t_3 represent the tangential deviations at A and D, for the tangent to the elastic curve at C; and let t_2 and t_4 represent the tangential deviations at C and B, for the tangent to the elastic curve at D; then

$$at_1 = -t_3$$

and

$$t_2 = -at_4$$

PQW is the M-diagram when AC is considered as a simple

span, to which the diagram $PUSV$ is added to provide for continuity.

$$EIt_1 = Pk(1-k)l\left(\tfrac{1}{2}l\right)\tfrac{2}{3}\left[kl + \tfrac{1}{2}(1-k)l\right] + M_1\left(\tfrac{1}{2}l\right)\left(\tfrac{2}{3}l\right)$$

$$= \frac{Pl^3}{6}(k - k^3) + \frac{M_1 l^2}{3}$$

$$EIt_3 = M_1\left(\tfrac{1}{2}al\right)\left(\tfrac{2}{3}al\right) + M_2\left(\tfrac{1}{2}al\right)\left(\tfrac{1}{3}al\right)$$

$$= \frac{a^2l^2}{6}(2M_1 + M_2)$$

$$EIt_2 = M_1\left(\tfrac{1}{2}al\right)\left(\tfrac{1}{3}al\right) + M_2\left(\tfrac{1}{2}al\right)\left(\tfrac{2}{3}al\right)$$

$$= \frac{a^2l^2}{6}(M_1 + 2M_2)$$

$$EIt_4 = M_2\left(\tfrac{1}{2}l\right)\left(\tfrac{2}{3}l\right) = \frac{M_2 l^2}{3}$$

Whence $$M_1 = \frac{-2Pl(k-k^3)(a+1)}{3a^2+8a+4}$$

$$M_2 = \frac{Pl(k-k^3)a}{3a^2+8a+4}$$

$$R_1 = P(1-k) - \frac{2P(k-k^3)(a+1)}{3a^2+8a+4}$$

$$R_2 = Pk + \frac{P(k-k^3)(2a^2+5a+2)}{(3a^2+8a+4)a}$$

$$R_3 = \frac{-P(k-k^3)(a^2+3a+2)}{(3a^2+8a+4)a}$$

$$R_4 = \frac{P(k-k^3)a}{3a^2+8a+4}$$

168. Three Equal Spans—Uniform Load.—The beam in Fig. 165 is continuous over three equal spans l, and supports a uniform load of w lb. per unit of length. Since the beam is symmetrical about the center, the ordinates in the M-diagram at B and C are equal; and only one elastic equation is necessary. Let the tangent to the elastic curve be drawn through B, and

let t_1 and t_3 represent the tangential deviations at A and C respectively; then

$$EIt_1 = \left(\frac{wl^2}{8}\right)\left(\frac{2}{3}l\right)\left(\frac{1}{2}l\right) + M\left(\frac{1}{2}l\right)\left(\frac{2}{3}l\right)$$
$$= \frac{wl^4}{24} + \frac{Ml^2}{3}$$

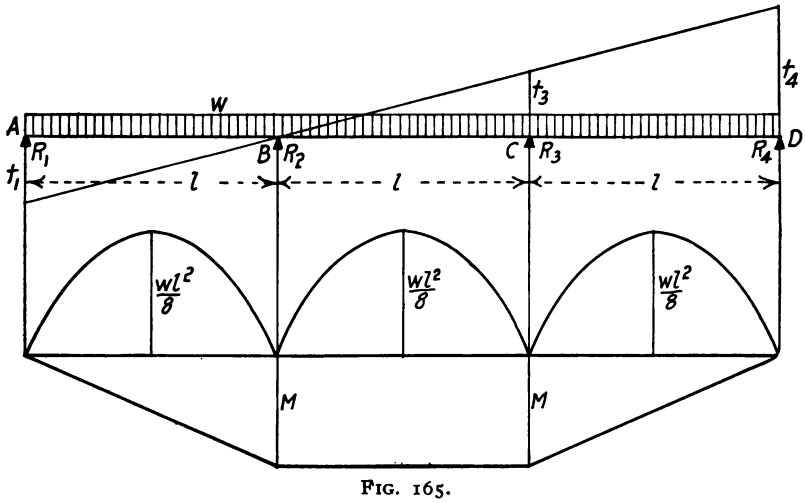


FIG. 165.

$$EIt_3 = \left(\frac{wl^2}{8}\right)\left(\frac{2}{3}l\right)\left(\frac{1}{2}l\right) + Ml\left(\frac{1}{2}l\right)$$
$$= \frac{wl^4}{24} + \frac{Ml^2}{2}$$
$$t_1 = -t_3$$
$$\frac{wl^4}{24} + \frac{Ml^2}{3} = -\frac{wl^4}{24} - \frac{Ml^2}{2}$$
$$M = -\frac{wl^2}{10}$$
$$R_1 = R_4 = \frac{4}{10}wl$$
$$R_2 = R_3 = \frac{11}{10}wl$$

If the M-diagram had not been symmetrical, either on account of unsymmetrical loading or variation in span lengths,

or both; the ordinates at B and C could not have been assumed equal. In this case there would have been two unknown ordinates, M_2 at B and M_3 at C; and two elastic equations would have been necessary for a solution.

These two equations may be written by establishing a relation between t_1 and t_3, and a relation between t_1 and t_4; or one tangent may be drawn through B and another through C.

169. Four Equal Spans—Uniform Load.—The beam in Fig. 166 is continuous over four equal spans l, and supports a uniform load of w lb. per unit of length. Since the beam is symmetrical about the center, there are only two unknown ordinates M_2 and M_3 to be determined, for which two elastic equations are necessary. On account of symmetry, the tangent to the elastic curve through C is horizontal, consequently the tangential deviations t_1 at A and t_2 at B are zero; hence

$$EIt_1 = \left(\frac{wl^2}{8}\right)\left(\frac{2}{3}l\right)\left(\frac{1}{2}l + \frac{3}{2}l\right) + M_2l^2 + M_3\left(\frac{1}{2}l\right)\left(\frac{5}{3}l\right) = 0$$

$$EIt_2 = \left(\frac{wl^2}{8}\right)\left(\frac{2}{3}l\right)\left(\frac{1}{2}l\right) + M_2\left(\frac{1}{2}l\right)\left(\frac{1}{3}l\right) + M_3\left(\frac{1}{2}l\right)\left(\frac{2}{3}l\right) = 0$$

or

$$6M_2 + 5M_3 = -wl^2$$

$$4M_2 + 8M_3 = -wl^2$$

$$M_2 = -\frac{3}{28}wl^2$$

$$M_3 = -\frac{2}{28}wl^2$$

Whence

$$R_1 = R_4 = \frac{11}{28}wl$$

$$R_2 = R_3 = \frac{32}{28}wl$$

If for any reason the M-diagram had not been symmetrical, the ordinates at B and D could not have been assumed equal; and the tangent to the elastic curve through C would not have been horizontal. There would have been three unknown ordinates, M_2 at B, M_3 at C and M_4 at D; and three elastic equations would have been necessary for a solution. These three equations may be written as follows: (a) draw a tangent

to the elastic curve at B and establish a relation between the tangential deviations at A and C; (b) draw a tangent to the elastic curve at C and establish a relation between the tangential deviations at B and D; (c) draw a tangent to the elastic curve at D and establish a relation between the tangential deviations at C and E.

The three equations may also be written as follows:

Let the tangent to the elastic curve be drawn through B; and let t_1, t_3, t_4 and t_5 represent the tangential deviations at A, C, D and E respectively. Then establish a relation be-

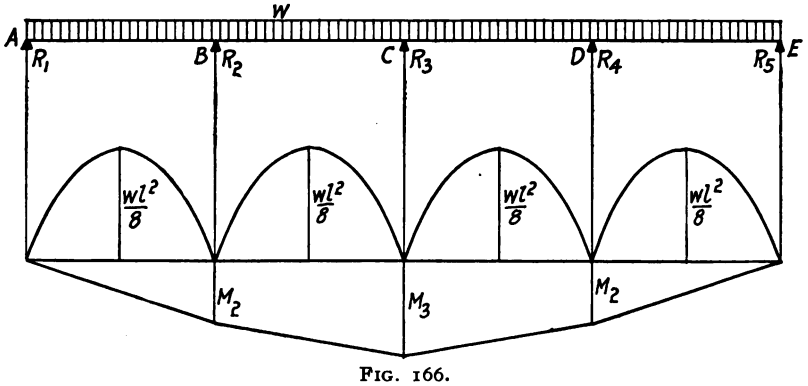


FIG. 166.

tween t_1 and t_3, a second relation between t_1 and t_4, and a third relation between t_1 and t_5.

In case the beam is restrained at each end by being fixed in a wall; the M-diagram presents five unknown ordinates, or one at each point of support. This condition requires two additional elastic equations, or five in all. These two equations may easily be written, since the tangents to the elastic curve through A and E are horizontal.

170. Coefficients for Pier Reactions.—When the spans are equal in length and the load is uniform throughout, the reaction at each support may be found by multiplying the total load on each span by the coefficients, as given in the following table. The Roman numerals represent the number of spans over which the beam is continuous.

REACTION COEFFICIENTS

I	$\frac{1}{2}$	$-\frac{1}{2}$				
II	$\frac{3}{8}$	$-\frac{10}{8}$	$-\frac{3}{8}$			
III	$\frac{4}{10}$	$-\frac{11}{10}$	$-\frac{11}{10}$	$-\frac{4}{10}$		
IV	$\frac{11}{28}$	$-\frac{32}{28}$	$-\frac{26}{28}$	$-\frac{32}{28}$	$-\frac{11}{28}$	
V	$\frac{15}{38}$	$-\frac{43}{38}$	$-\frac{37}{38}$	$-\frac{37}{38}$	$-\frac{43}{38}$	$-\frac{15}{38}$

SEC. III. CLAPEYRON'S THEOREM OF THREE MOMENTS

171. Clapeyron's theorem may be used to establish a relation between the bending moments at any three consecutive supports of a beam of uniform cross-section, as shown in Fig. 167. Let l_1 and l_2 represent the lengths of any two adjacent spans, which support the uniform loads of w_1 and w_2 per unit of length respectively; and let M_0, M_1 and M_2 represent the bending moments at the three supports. Let the tangent to the elastic curve be drawn through the middle support; and let t_0 and t_2 represent the tangential deviations at the left and right supports respectively; then

$$EIt_0 = \left(\frac{w_1 l_1^2}{8}\right)\left(\frac{2}{3}l_1\right)\left(\frac{1}{2}l_1\right) + M_0\left(\frac{1}{2}l_1\right)\left(\frac{1}{3}l_1\right) + M_1\left(\frac{1}{2}l_1\right)\left(\frac{2}{3}l_1\right)$$

$$= \frac{w_1 l_1^4}{24} + \frac{M_0 l_1^2}{6} + \frac{M_1 l_1^2}{3}$$

$$EIt_2 = \left(\frac{w_2 l_2^2}{8}\right)\left(\frac{2}{3}l_2\right)\left(\frac{1}{2}l_2\right) + M_1\left(\frac{1}{2}l_2\right)\left(\frac{2}{3}l_2\right) + M_2\left(\frac{1}{2}l_2\right)\left(\frac{1}{3}l_2\right)$$

$$= \frac{w_2 l_2^4}{24} + \frac{M_1 l_2^2}{3} + \frac{M_2 l_2^2}{6}$$

$$t_0 l_2 = -t_2 l_1$$

Whence

$$M_0 l_1 + 2M_1(l_1 + l_2) + M_2 l_2 = -\frac{1}{4}(w_1 l_1^3 + w_2 l_2^3) \qquad (14)$$

In any continuous beam, not restrained at the ends having n spans, there are $n-1$ unknown bending moments at the intermediate supports, which may be determined by writing

$n-1$ equations similar to (14). Equations of this type are applicable when any span, taken at random, supports either a uniform load over its entire length, or no load whatever. When two adjacent spans have the same length l, and support the same load w per unit of length, Eq. (14) reduces to

$$M_0 + 4M_1 + M_2 = -\frac{1}{2}wl^2$$

An application of Eq. (14) will be made in connection with

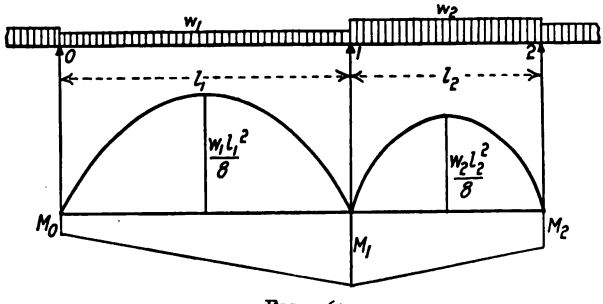


FIG. 167.

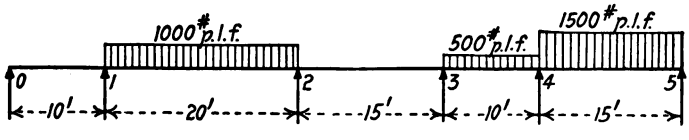


FIG. 168.

the continuous beam of uniform cross-section, having five spans (Fig. 168). First and second spans:

$$10M_0 + 2M_1(10 + 20) + 20M_2 = -\frac{1}{4}(0 + 1{,}000 \times 20^3)$$

Second and third spans:

$$20M_1 + 2M_2(20 + 15) + 15M_3 = -\frac{1}{4}(1{,}000 \times 20^3 + 0)$$

Third and fourth spans:

$$15M_2 + 2M_3(15 + 10) + 10M_4 = -\frac{1}{4}(0 + 500 \times 10^3)$$

Fourth and fifth spans:

$$10M_3 + 2M_4(10 + 15) + 15M_5 = -\frac{1}{4}[(500 \times 10^3) + (1{,}500 \times 15^3)]$$

Since the beam is simply supported at each end,

$$M_0 = 0$$
$$M_5 = 0$$

A solution of these equations will give the bending moments at the points of support, and the reactions may be determined from the principles of statics.

SEC. IV. PARTIALLY CONTINUOUS BEAMS

172. No Shear Transmitted.—It is frequently desirable to consider a structure in which the continuity is imperfect. A swing truss bridge on four supports, designed with parallel

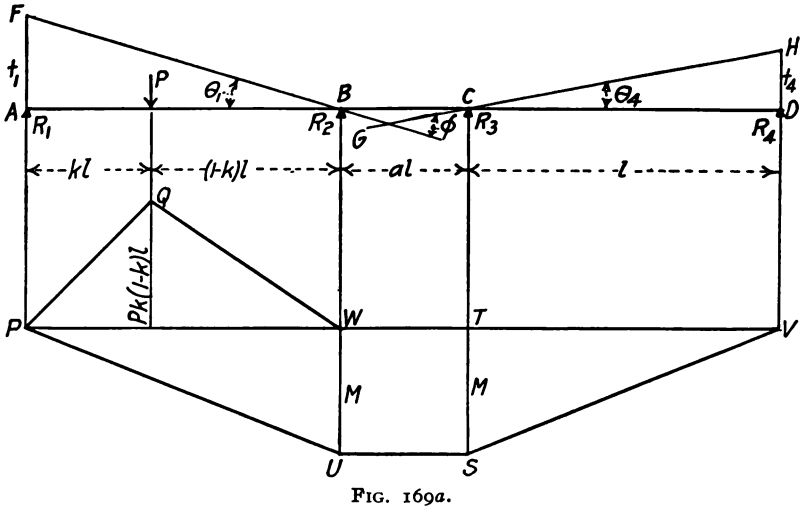


FIG. 169*a*.

chords and very light web members in the center span, so that no shear can be transmitted between the two inside supports is a structure of this kind. Such structures are called *partially continuous*, and their treatment will be illustrated by the beam in Fig. 169*a*.

It is assumed that bending moment, but no shear exists in the center span; hence $R_3 = -R_4$, and the bending moment M at B equals the bending moment at C. Since the continuity of the beam is broken at B and C, the elastic curve is not continuous, but forms cusps at these points; and the tangent FG

to the elastic curve for AB at B is *not* tangent to the elastic curve for BC. Similarly the tangent HG to the elastic curve for CD at C is *not* tangent to the elastic curve for BC. Let θ_1 = angle ABF, and θ_4 = angle DCH, then

$$\theta_1 + \theta_4 = \phi = \frac{area\ WTSU}{EI}$$

The tangential deviations at A and D, being represented as measured above the axis of the beam, are considered negative.

$$-t_1 = \theta_1 l$$

$$-t_4 = \theta_4 l$$

$$t_1 + t_4 = -\phi l$$

$$EI\phi = Mal$$

$$EIt_1 = Pk(1-k)l\left(\tfrac{1}{2}l\right)\tfrac{2}{3}\left[kl + \tfrac{1}{2}(1-k)l\right] + M\left(\tfrac{1}{2}l\right)\left(\tfrac{2}{3}l\right)$$

$$= \frac{Pl^3}{6}(k - k^3) + \frac{Ml^2}{3}$$

$$EIt_4 = \frac{Ml^2}{3}$$

Since $EIt_1 + EIt_4 = -EI\phi l$

Then
$$\frac{Pl^3}{6}(k - k^3) + \frac{Ml^2}{3} + \frac{Ml^2}{3} = -Mal^2$$

or
$$M = -\frac{Pl(k - k^3)}{4 + 6a}$$

Therefore
$$R_1 = P(1-k) - \frac{P(k - k^3)}{4 + 6a}$$

$$R_2 = Pk + \frac{P(k - k^3)}{4 + 6a}$$

$$R_3 = \frac{P(k - k^3)}{4 + 6a}$$

$$R_4 = -\frac{P(k - k^3)}{4 + 6a}$$

173. No Moment Transmitted.—The span in Fig. 169*b* consists of two restrained beams, connected at mid-span in such a way that shear, but no bending moment, can be transmitted from one beam to the other. The span therefore represents a different phase of partial continuity from that of the previous problem. The principle here involved is employed in the

design of a bascule span composed of two leaves connected by a shear lock. The principle must be modified, however, in its application to a bascule span; for the leaves do not as a rule have a constant moment of inertia, nor are they in perfect restraint at the points of support.

A constant moment of inertia and perfect restraint will be assumed in finding the shear V on the pin-connection at C,

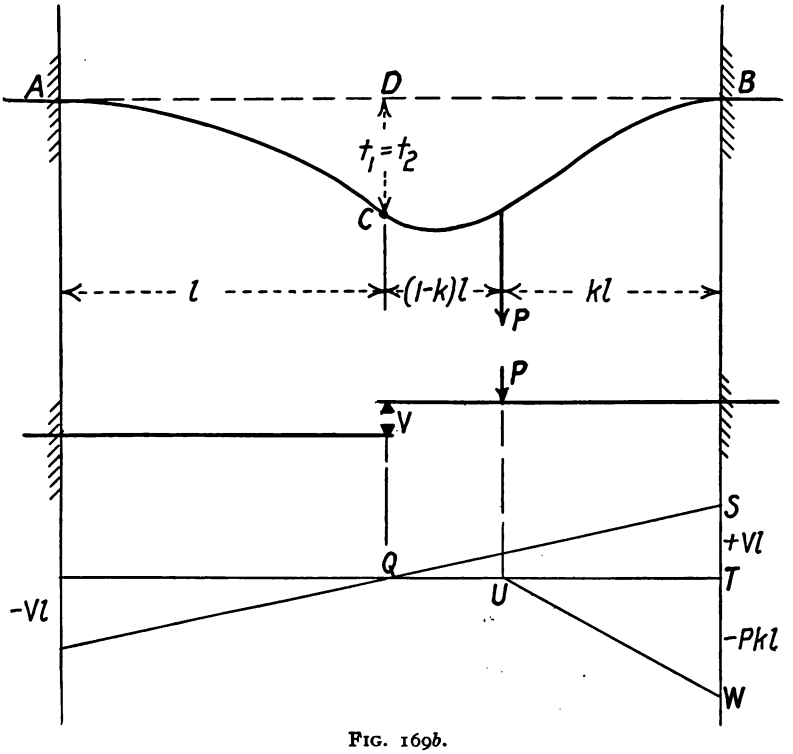


FIG. 169b.

when the beam CB supports the load P as shown. The M-diagram may be drawn very easily when the partially continuous beam ACB is considered as two restrained beams sketched separately; with the shear at C considered as a force V, acting upward on CB and downward on CA. The bending moment at C is zero. The M-diagram for CB is best sketched in two parts—the area QST representing the bending moment of V, and the area TUW representing the bending moment of P.

Since the continuity is broken at C, it cannot be assumed that the total area of the M-diagram is zero, although the angle ϕ between AD and BD is zero. The absurdity of such an assumption is obvious when $k = 1$ and the load P is at C, in which case it is clear that the M-diagram is a negative area throughout and cannot equal zero; neither can the tangential deviation at either A or B be equated to zero, for a similar reason.

Let t_1 represent the tangential deviation for the beam AC and t_2 for the beam BC, then

$$t_1 = t_2$$

$$EIt_1 = (-Vl)\left(\tfrac{1}{2}l\right)\left(\tfrac{2}{3}l\right) = -\frac{Vl^3}{3}$$

$$EIt_2 = (Vl)\left(\tfrac{1}{2}l\right)\left(\tfrac{2}{3}l\right) - (Pkl)\left(\tfrac{1}{2}k\right)l\left(l - \tfrac{1}{3}kl\right)$$

Whence $V = \frac{P}{4}(3k^2 - k^3)$

SEC. V. CONTINUOUS BEAMS IN FOUNDATIONS

174. In designing foundations for high buildings, it is frequently necessary to place three columns on a single footing. An attempt is made to secure uniform soil pressure by supporting the columns on longitudinal girders resting on shorter cross-beams. These beams bear on the soil and distribute the column loads over a sufficient area of foundation. In the design of such a foundation the engineer frequently adopts the following course. Having determined the sum total of the three column loads, he decides upon the allowable unit bearing pressure of the soil; determines the required area of the foundation; and then shapes the footing by assuming arbitrarily either the length or width of the footing to suit his convenience. The last step in this process disregards the fact that the girders and their loading constitute a statically indeterminate system. The arbitrary proportioning of such a system does not result in uniform soil pressure; hence the actual soil pressure will be greater at some points; and less at other points, than the allowable pressure.

Two difficulties are encountered in making a rational analysis of the problem: (1) The columns are usually anchored to the

girders with sufficient rigidity to allow for the transference of moment. In the discussion which follows, this moment will be neglected. (2) Since the girders distort slightly in performing their office of distributing the loads, the assumption of uniformly distributed soil bearing pressure, which the designer necessarily makes (in the present analysis as well as in the customary procedure) is unavoidably vitiated. There is no data available as to the importance of this effect; but it seems reasonable to conclude that a state of uniform pressure distribution is more nearly approximated when the problem is solved by an analysis that takes account of the continuity of the girders, than by the present arbitrary choice of proportions.

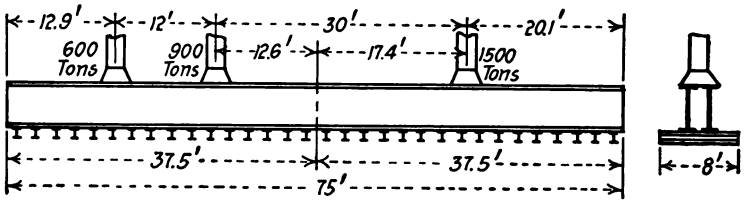


FIG. 170.

The arbitrary method of proportioning will be given in connection with Fig. 170. The sum of the three column loads is 3,000 tons. If a permissible soil reaction of 5 tons per square foot is assumed, the required area of the footing is 600 sq. ft. The size of the footing is taken arbitrarily as 75 ft. long and 8 ft. wide. The center of gravity of the three column loads is 12.6 ft. to the right of the center column, which locates the center of the soil reaction; hence the girders should extend 12.9 ft. beyond the left column and 20.1 ft. beyond the right column.

A grillage so designed conforms to the three conditions of static equilibrium—*i.e.* (sum of the vertical forces equals zero, sum of the horizontal forces equals zero, and the sum of the moments about any point equals zero). From the foregoing, it is concluded that the three concentrated column loads are supported by a uniform soil reaction of 5 tons per square foot or 40 tons per linear foot of girder. That this conclusion is fallacious, however, becomes apparent when the structure

(Fig. 170) is inverted (Fig. 171); and the soil pressure represented as a uniform load of 40 tons per linear foot resting upon three supports. It is therefore evident that the problem is statically indeterminate, and the odds are greatly against the probability that the reactions are 600, 900 and 1,500 tons respectively.

The rational method takes account of the continuity of the longitudinal girders. Three independent simultaneous equations are required for the determination of the three reactions (Fig. 171). Two of these equations $\Sigma V = 0$ and $\Sigma M = 0$ are

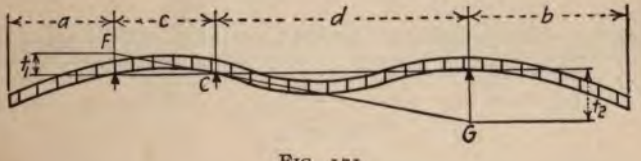


FIG. 171.

supplied by the principles of statics. The third or elastic equation is derived by drawing the tangent *FG* through *C*, and establishing the relation between t_1 and t_2. The application of these three equations to practical cases differs in detail, according to the physical conditions governing the problem. Three of several possible cases are illustrated in Figs. 172, 173 and 174, and will be considered separately. In each case the following points are to be observed: Each figure is shown inverted for convenience. The known column reactions are represented by *P*, *Q* and *R*. The spacing of the columns, being fixed by the architectural features, is known; and represented by *c* and *d*. The tangential deviations t_1 and t_2, not shown in Figs. 172, 173 and 174, are to be taken as represented in Fig. 171. Three quantities, differing in each case, are to be determined by a solution of the three independent simultaneous equations cited.

175. Case I. *Projections Not Limited by Site.*—In Fig. 172, let *w* represent the intensity of the uniform soil pressure in pounds per linear foot. If the architectural features do not limit the end projections *a* and *b* of the main grillage girders; it will be possible to attain this condition of uniform soil

pressure by selecting a, b and w accordingly. The two static equations are

$$\Sigma V = P + Q + R - (a + b + c + d)w = 0 \tag{15}$$

$$\Sigma M = aP + (a + c)Q + (a + c + d)R - \frac{w}{2}(a + b + c + d)^2 = 0 \tag{16}$$

The elastic equation is

$$\frac{t_1}{t_2} = -\frac{c}{d} \tag{17}$$

The expressions for t_1 and t_2 may be determined as follows:

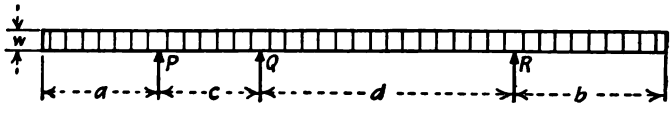


FIG. 172.

The bending moment between P and Q at any distance x from P is

$$M_1 = Px - \frac{1}{2}w(a + x)^2$$

$$\text{hence} \quad EIt_1 = \int_o^c M_1 x dx = \frac{c^3 P}{3} - \frac{w}{2}\left(\frac{a^2c^2}{2} + \frac{2ac^3}{3} + \frac{c^4}{5}\right)$$

Similarly, the bending moment between Q and R at any distance x from R is

$$M_2 = Rx - \frac{1}{2}w(b + x)^2$$

$$\text{hence} \quad EIt_2 = \int_o^d M_2 x dx = \frac{d^3 R}{3} - \frac{w}{2}\left(\frac{b^2d^2}{2} + \frac{2bd^3}{3} + \frac{d^4}{4}\right)$$

Substituting the values of t_1 and t_2 in Eq. (17) and reducing

$$\frac{8c^3P - w(6a^2c^2 + 8ac^3 + 3c^4)}{8d^3R - w(6b^2d^2 + 8bd^3 + 3d^4)} = -\frac{c}{d} \tag{17a}$$

Eliminating w from Eqs. (15), (16) and (17a)

$$b = a + \frac{c(-P + Q + R) + d(-P - Q + R)}{P + Q + R} \tag{18}$$

$$6c(P + Q + R)a^2 + 6d(P + Q + R)b^2 + 8(c^2Q + c^2R - d^2R)a + 8(d^2P + d^2Q - c^2P)b + (-5c^3 + 3d^3 - 8c^2d)P + 3(c^3 + d^3)Q + (-5d^3 + 3c^3 - 8cd^2)R = 0 \tag{19}$$

Illustrative Problem.—Let $c = 12$ ft., $d = 30$ ft., $P = 600$ tons, $Q = 900$ tons and $R = 1{,}500$ tons.

Substituting the numerical values in Eqs. (18) and (19), and reducing

$$b = a + 7.2$$

$$10a^2 + 25b^2 - 372a + 468b = 10{,}374$$

whence $a = 7.8$ ft., and $b = 15$ ft.

The length of the base is $42 + 22.8$ ft. or 64.8 ft., and the soil pressure per linear foot is

$$w = \frac{3{,}000}{64.8} = 46.3 \text{ tons.}$$

If the allowable bearing pressure on the soil is 5 tons per square foot, the foundation should have a width of

$$\frac{46.3}{5} = 9.5 \text{ ft.}$$

Thus, in Figs. 170 or 172, the girders should be about 65 ft. long; extending approximately 8 ft. beyond the left column and 15 ft. beyond the right column. The beams in the lower tier of grillage should have a length of about 9.5 ft.

176. Case II. *Projection at One End Limited by Site.*—Architectural features frequently fix the length to which the footing

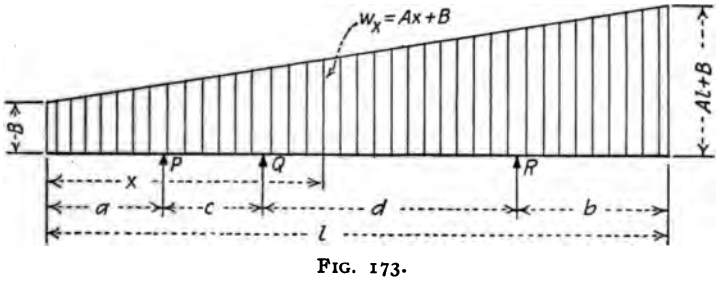


FIG. 173.

may extend beyond one of the end columns. Sometimes it is necessary to allow no extension whatever at one end. Whenever either of these two limitations arises, the footing may be so arranged that its pressure per foot of length varies uniformly from one to the other; as shown in Fig. 173. This may be accomplished by a variation in the lengths of the cross-beams, resulting in a trapezoidal area for the footing. Care should be

exercised in the choice and spacing of the cross-beams in order that equal deflections at their centers may be assured.

Let the intensity of the soil pressure per linear foot, at a distance x from the left end be,

$$w_x = Ax + B$$

so that the soil pressure at the left end will have the intensity B per linear foot, and at the right end the intensity $Al + B$ per linear foot.

The static equations are

$$\Sigma V = Al^2 + 2Bl - 2(P + Q + R) = 0 \qquad (20)$$

$$\Sigma M = 2Al^3 + 3Bl^2 - 6aP - 6(a + c)Q - 6(a + c + d)\,R = 0 \qquad (21)$$

The elastic Eq. (17), when developed, may be written in the form

$$\begin{aligned} &c[4(a + c)^3 + a\{3(a + c)^2 + a(3a + 2c)\}]A \\ &+ d[(a + c + d)^3 + (a + c)\{2(a + c + d)^2 + (a + c) \\ &\qquad [7(a + c) + 3d]\}]A \\ &+ 5c[3(a + c)^2 + a(3a + 2c)]B + 5d[(a + c + d)^2 + (a + c) \\ &\qquad \{5(a + c) + 2d\}]B \\ &= 20[(c + d)^2 + c(c + d)]P + 20d^2Q \end{aligned} \qquad (22)$$

Three of the four quantities a, b, A or B may be determined from these equations. A numerical value for either a or b may be chosen arbitrarily to suit the architectural features, and the other three quantities determined by a solution of Eqs. (20), (21) and (22).

Illustrative Problem.

Let $c = 12$ ft., $d = 30$ ft., $P = 600$ tons, $Q = 900$ tons and $R = 1{,}500$ tons. The architectural features do not allow a to exceed 6 ft., and the designer wishes to utilize this full amount. In other words, he fixes $a = 6$ ft.; and determines b, A and B, from the equations. Substituting the numerical values in (20), (21) and (22)

$$Al^2 + 2Bl = 6{,}000$$

$$2Al^3 + 3Bl^2 = 550{,}800$$

$$8{,}273{,}664A + 824{,}040B = 43{,}416{,}000$$

whence $$l^3 - 227.761l^2 + 13{,}884.448l - 209{,}928.788 = 0$$

or $$l = 22.78 \text{ or } 66.58 \text{ or } 138.4 \text{ ft.}$$

If 66.58 ft. is taken, then

$$A = -0.328$$
$$B = 55.98$$

and $$Al + B = 34.13$$

Hence the intensity of the soil pressure is 55.98 tons per linear foot at the left end; and decreases uniformly to 34.13 tons at the right end. Since $a = 6$ ft., and $l = 66.58$ ft., the foundation extends $b = 18.58$ ft. beyond the right column.

Since the quantity b does not appear in Eqs. (20), (21) and (22), it will be found advantageous in any numerical problem to arrange the nomenclature so that P may represent the end column beyond which the length of the foundation is limited; and assign a numerical value to a. When no extension of the foundation beyond P is allowed, $a = 0$.

177. Case III. *Both Projections Limited by Site.*—In the case illustrated by Fig. 174, where both a and b are fixed or equal

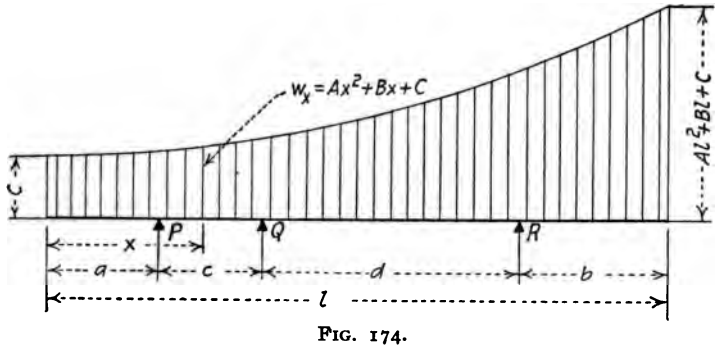


FIG. 174.

zero; the footing may be so proportioned as to produce an intensity of soil pressure per linear foot, which varies according to the ordinates of a parabolic curve. The soil pressure per square foot may be made uniform by varying the length of the cross-beams accordingly. The dimensions a, b, c and d, and the loads P, Q and R will be known; while the three constants which determine the intensity of the soil pressure are to be found by solution.

Let the intensity of the soil pressure per linear foot at a distance x from the left end be

$$w_x = Ax^2 + Bx + C$$

so that the soil pressure at the left end will have the intensity C per linear foot, and at the right end the intensity $Al^2 + Bl + C$ per linear foot.

The static equations are

$$\Sigma V = 2Al^3 + 3Bl^2 + 6Cl - 6(P + Q + R) = 0 \qquad (23)$$

$$\Sigma M = 3Al^4 + 4Bl^3 + 6Cl^2 - 12aP - 12(a + c)Q - 12(a + c + d)R = 0 \qquad (24)$$

The elastic Eq. (17) when developed, may be written in the form

$$\begin{aligned}
&c[5(a + c)^4 + a\{4(a + c)^3 + a[3(a + c)^2 + a(3a + 2c)]\}]A \\
&+d[(a + c + d)^4 + (a + c)\{2(a + c + d)^3 + \\
&\qquad (a + c)[3(a + c + d)^2 + (a + c)\{9(a + c) + 4d\}]\}]A \\
&+3c[4(a + c)^3 + a\{3(a + c)^2 + a(3a + 2c)\}]B \\
&+3d[(a + c + d)^3 + (a + c)\{2(a + c + d)^2 + \\
&\qquad (a + c)[7(a + c) + 3d]\}]B \\
&+[15c\{3(a + c)^2 + a(3a + 2c)\} + 15d\{(a + c + d)^2 + \\
&\qquad (a + c)[5(a + c) + 2d]\}]C \\
&= 60[(c + d)^2 + c(c + d)]P + 60d^2Q
\end{aligned} \qquad (25)$$

Illustrative Problem.

Let $a = 6$ ft., $b = 12$ ft., $c = 12$ ft., $d = 30$ ft., $P = 600$ tons, $Q = 900$ tons and $R = 1{,}500$ tons. Then $l = 60$ ft. Substituting the numerical values in Eqs. (23), (24) and (25), and reducing

$$\begin{aligned}
1{,}200A + 30B + C &= 50 \\
1{,}800A + 40B + C &= 51 \\
623{,}028A + 38{,}304B + 3{,}815C &= 201{,}000
\end{aligned}$$

whence

$$\begin{aligned}
A &= \ \ 0.0291 \\
B &= -1.646 \\
C &= \ \ 64.466
\end{aligned}$$

and the intensity of the soil pressure per linear foot at any distance x from the left end is

$$w_x = 0.0291x^2 - 1.646x + 64.466$$

The intensities of the soil pressure per linear foot at the two ends and at five intermediate points are

x Feet	w Tons per linear foot	x Feet	w Tons per linear foot
0	64.5	40	45.0
10	51.0	50	55.0
20	43.0	60	70.5
30	41.0		

If the footing is to extend a given distance, say 5 ft. beyond one end column, and is not to extend beyond the other end column, it will be found advantageous to arrange the nomenclature so that $a = 0$ and $b = 5$. Equations (23), (24) and (25) are applicable when the footing is not to extend beyond either end column; in which case $a = 0$ and $b = 0$, are to be substituted.

CHAPTER VII

DEFLECTION OF TRUSSES

178. Stress and Strain.—The stress in any member of a truss under the influence of external forces is accompanied by a corresponding strain or change in length of the member. The strains which the various members experience when under stress, cooperate in a general distortion of the truss; somewhat after the manner illustrated in Fig. 175. The full lines represent the configuration of the truss when the members are under no strain. When loads are applied gradually at B, C and G, the truss undergoes a gradual distortion; and finally conforms to the shape indicated by the dotted lines, when the loads have reached their full magnitudes F_1, F_2 and F_3. The dotted outline gives an exaggerated idea of the distortion which may be expected in any practical problem.

The first step in finding the movement of displacement of any point in the truss, is the determination of the strain in each member subjected to stress. This is easily accomplished by the well established law of mechanics, *viz.:*

$\frac{\text{unit stress}}{\text{unit strain}}$ = modulus (measure or coefficient) of elasticity.

Let P = the stress in a member in pounds.

A = the area of cross-section of the member in square inches (inch^2).

l = the length of the member in inches.

D = strain, or change in length, of the member in inches.

$\delta = \frac{D}{l}$ = unit strain in inches per inch (inch/inch), a ratio.

$\frac{P}{A}$ = unit stress in pounds per square inch (pounds/inches^2).

E = modulus of elasticity in pounds per square inch (pounds/inches.^{2})

then $\frac{\frac{P}{A}}{\frac{D}{l}} = E$ or $D = \frac{Pl}{AE}$

Hence the strain or change in length of a member may be determined if its length and cross-sectional area, the stress which it resists and the modulus of elasticity of its material are known. Experiments show that, for any given material, the modulus of elasticity is approximately constant for all unit stresses below a certain limit, called the elastic limit. The elastic limit for structural steel is about 60 per cent of its ultimate strength; hence the permissible unit stress in all current practice is well within the elastic limit. The modulus of elasticity for ordinary structural steel is about 29,000,000 lb. per square inch; hence if a member 50 ft. long, having a cross-sectional area of 20 sq. in., is subjected to a tensile stress of 300,000 lb., the strain or elongation of the member will be

$$D = \frac{300{,}000 \times 50 \times 12}{20 \times 29{,}000{,}000} = 0.31 \text{ in.}$$

SEC. I. ALGEBRAIC METHOD

179. The algebraic solution may be developed by equating the external work done by the external forces, to the internal

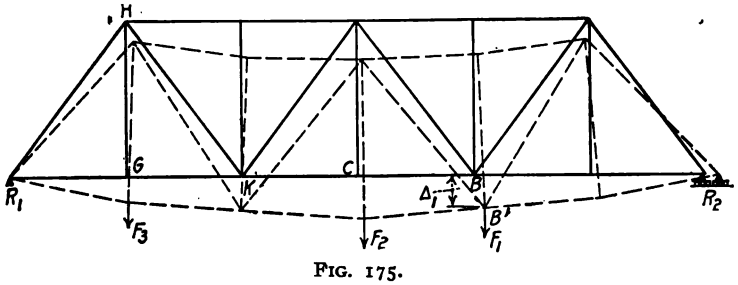


FIG. 175.

work performed upon the members of the truss.

Let Δ_1 (Fig. 175) represent the vertical component of the displacement, or deflection of the point B in the direction of the force F_1. During the movement of the point from B to B', the force which has been increased from zero to F_1 performs work to the amount of $\frac{1}{2} F_1\Delta_1$. Each member under stress, and thereby subject to strain, contributes its share to the deflection of B. Consider any member HK, for example, which is l inches long and has a cross-sectional area of A sq. in. Let P_1, P_2 and P_3

represent the stresses in HK, caused by the loads F_1, F_2 and F_3 respectively; and let $P_1 + P_2 + P_3 = P =$ the total stress. The strain in HK is $D = \frac{Pl}{AE}$

and the total work performed upon HK, as the three loads are applied and the stresses gradually increased from zero to their final values, is

$$\tfrac{1}{2}(P_1 + P_2 + P_3)D = \tfrac{1}{2}PD = \tfrac{1}{2}P\frac{Pl}{AE}$$

The work performed upon HK, resulting from the load F_1 is

$$\tfrac{1}{2}P_1D = \tfrac{1}{2}P_1\frac{Pl}{AE}$$

Let $\Sigma\ \frac{1}{2}P_1\frac{Pl}{AE}$ represent the sum of the work performed upon each member by the stress resulting from the load F_1. Since the external work done by the loads must equal the internal work performed upon the members, then

$$\tfrac{1}{2}F_1\,\Delta_1 = \Sigma\tfrac{1}{2}P_1\frac{Pl}{AE}$$

Whence

$$\Delta_1 = \Sigma\frac{P_1}{F_1}\frac{Pl}{AE}$$

in which P_1 is the stress in any member due to the load F_1, and P is the stress in the same member due to the three loads F_1, F_2 and F_3.

Since the stress P_1 in any member varies directly as the load F_1; it is obvious that the ratio $\frac{P_1}{F_1}$ is constant for that member for all values of F_1.

Let

$$\frac{P_1}{F_1} = u.$$

or

$$\Delta_1 = \Sigma\frac{Pul}{AE} \qquad (1)$$

To get the value of the ratio u for each member, assume for the present that F_1 equals 1 lb.; in other words, place 1 lb. at the point, acting in the direction of the desired deflection and compute the stresses in all members for this 1 lb. loading. The resulting stresses expressed in pounds, when divided by 1 lb., will represent the ratio u for the various members. For

any member, the stress P and the ratio u have positive or negative signs corresponding to tension or compression. The strain in any member causes the point in question to deflect in the direction in which the 1 lb. load is assumed to act, if P and u have like signs; and in the opposite direction, if they have unlike signs.

In Eq. (1) E is the only quantity which is constant for all members of the truss, provided all members are made of the same material (which is usually the case). Hence the expression $\frac{Pul}{A}$ may be tabulated and computed for each member; and the deflection found by dividing the algebraic sum $\Sigma \frac{Pul}{A}$ by E, as in the following problem:

180. Illustrative Problem.

The truss in Fig. 176 supports a load of 240,000 lb. at each bottom chord panel point. In Table I, the length of each mem-

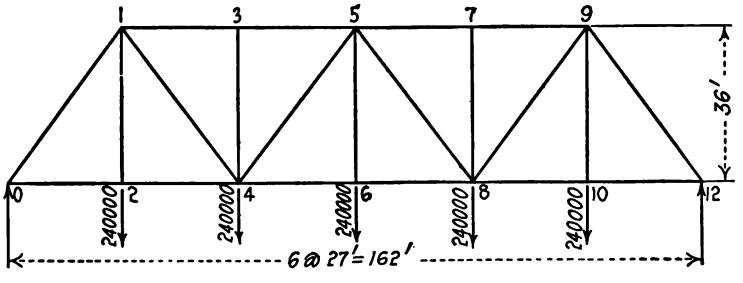


FIG. 176.

ber is given in column 1; the *gross* cross-sectional, in column 2; and the stress, in column 3. The quantities in column 4, when divided by E, represent the strains. To find the deflection at point 6, place 1 lb. at 6 and compute the stresses; as given in Fig. 177. These stresses, when divided by 1 lb., are the ratios u_6 in column 5. In this case P and u_6 have like signs for each member, hence the quantities in column 6 are all positive, and their algebraic sum is 42,246. Since P was expressed in units of 1,000 lb., the deflection at point 6 is

$$\Delta_6 = \frac{42{,}246 \times 1{,}000}{E} = \frac{42{,}246{,}000}{29{,}000{,}000} = 1.46 \text{ in.}$$

TABLE I

Member	1	2	3	4	5	6		7	8		9	10
	Length inches l	Area square inches A	Stress in thousands of pounds P	$\frac{Pl}{A}$	u_6	$\frac{Pul}{A}$	Member	u_4	$\frac{Pul}{A}$	Member	u_5	$\frac{Pul}{A}$
0–1	540	84.0	−750	−4,820	$-\frac{5}{8}$	+3,013	0–1	$-\frac{5}{8}$	+4,017	0–1	$-\frac{25}{32}$	+5,021
1–3	324	69.0	−720	−3,380	$-\frac{3}{4}$	+2,540	1–3	−1	+3,380	1–3	$-\frac{1}{2}$	+1,690
3–5	324	69.0	−720	−3,380	$-\frac{3}{4}$	+2,540	3–5	−1	+3,380	3–5	$-\frac{1}{2}$	+1,690
0–2	324	47.4	+450	+3,080	$+\frac{3}{8}$	+1,155	0–2	$+\frac{1}{2}$	+1,540	0–2	$+\frac{3}{8}$	+1,925
2–4	324	47.4	+450	+3,080	$+\frac{3}{8}$	+1,155	2–4	$+\frac{1}{2}$	+1,540	2–4	$+\frac{3}{8}$	+1,925
4–6	324	83.4	+810	+3,150	$+\frac{9}{16}$	+3,540	4–6	$+\frac{3}{4}$	+2,360	4–6	$+\frac{9}{16}$	+1,180
1–4	540	47.4	+450	+5,125	$+\frac{5}{8}$	+3,200	1–4	$+\frac{5}{6}$	+4,270	1–4	$-\frac{5}{64}$	−1,070
4–5	540	30.2	−150	−2,680	$-\frac{5}{6}$	+1,675	4–5	$+\frac{5}{12}$	−1,120	4–5	$+\frac{5}{64}$	− 560
1–2	432	22.5	+240	+4,610	0	0	1–2	0	0	1–2	+1	+4,610
3–4	432	22.5	0	0	0	0	3–4	0	0	3–4	0	0
5–6	432	22.5	+240	+4,610	+1	+4,610	5–6	0	0	5–6	0	0
7–8	432	22.5	0	0	0	0	7–8	0	0	7–8	0	0
9–10	432	22.5	+240	+4,610	0	0	9–10	0	0	9–10	0	0
5–8	540	30.2	−150	−2,680	$-\frac{5}{6}$	+1,675	5–8	$-\frac{5}{12}$	+1,120	5–8	$-\frac{5}{64}$	+ 560
8–9	540	47.4	+450	+5,125	$+\frac{5}{8}$	+3,200	8–9	$+\frac{5}{12}$	+2,140	8–9	$+\frac{5}{64}$	+1,070
6–8	324	83.4	+810	+3,150	$+\frac{9}{16}$	+3,540	6–8	$+\frac{3}{4}$	+2,360	6–8	$+\frac{9}{16}$	+1,180
8–10	324	47.4	+450	+3,080	$+\frac{3}{8}$	+1,155	8–10	$+\frac{1}{6}$	+ 770	8–10	$+\frac{1}{8}$	+ 385
10–12	324	47.4	+450	+3,080	$+\frac{3}{8}$	+1,155	10–12	$+\frac{1}{6}$	+ 770	10–12	$+\frac{1}{8}$	+ 385
5–7	324	69.0	−720	−3,380	$-\frac{3}{4}$	+2,540	5–7	$-\frac{1}{2}$	+1,690	5–7	$-\frac{1}{4}$	+ 845
7–9	324	69.0	−720	−3,380	$-\frac{3}{4}$	+2,540	7–9	$-\frac{1}{2}$	+1,690	7–9	$-\frac{1}{4}$	+ 845
9–12	540	84.0	−750	−4,820	$-\frac{5}{8}$	+3,013	9–12	$-\frac{5}{12}$	+2,009	9–12	$-\frac{5}{64}$	+1,004
						$\Sigma\frac{Pul}{A}$ = 42,246			+33,036			+24,315
									− 1,120			− 1,630
									$\Sigma\frac{Pul}{A}$ = 31,916			$\Sigma\frac{Pul}{A}$ = 22,685

In like manner the deflection at point 4 is found by placing a load of 1 lb. at 4 (Fig. 178) and obtaining the ratios u_4 in column 7. The products of corresponding values in columns 4 and 7 are given in column 8. In this instance, P and u_4 have unlike signs for the member 4–5; hence the quantity representing the member 4–5 in column 8 has the negative sign, and the strain in that member tends to raise the point 4. The alge-

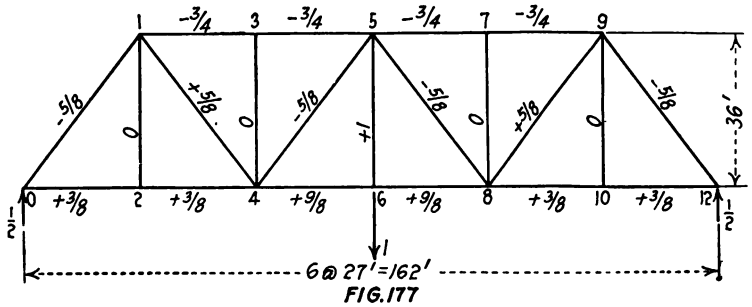


FIG. 177

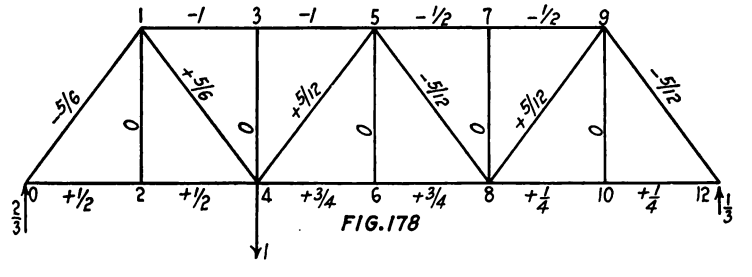


FIG. 178

braic sum of the quantities in column 8 is 31,916; and the deflection of point 4 is

$$\Delta_4 = \frac{31{,}916{,}000}{29{,}000{,}000} = 1.1 \text{ in.}$$

Similarly the deflection of point 6 is

$$\Delta_6 = \frac{22{,}685{,}000}{29{,}000{,}000} = 0.78 \text{ in.}$$

Since the design and loading of the truss in this problem are both symmetrical about the center line, it is obvious that the vertical deflections at the points 8 and 10 are respectively the same as at points 4 and 2.

The horizontal displacement of any point may be obtained

in a similar manner. Suppose that the horizontal displacement of the point 1 in Fig. 176 is required, when the truss is held fast at the left support and rests on rollers at the right support; as shown in Fig. 175. The loads are assumed, as in Fig. 176; and the values of $\frac{Pl}{A}$ of column 4, Table I, are therefore applicable. Place 1 lb. at point 1, acting horizontally either to the right or left, let us say to the right (as in Fig. 179); compute

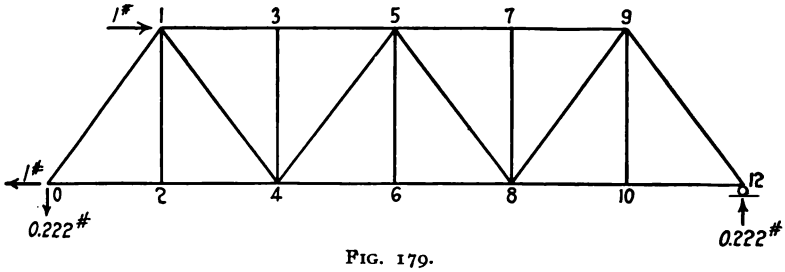


FIG. 179.

the reactions and tabulate the resulting ratios in a column marked u_1; and make the extensions $\frac{Pu_1l}{A}$.

The horizontal displacement of the point 1 will be

$$\Delta_1 = \Sigma \frac{Pu_1l}{AE}$$

Since the 1 lb. load was taken as acting to the right, a positive value for the summation will indicate a displacement to the right; and *vice versa*.

SEC. II. GRAPHIC METHOD

181. Williot Diagrams.—The graphic method of finding the deflections of a truss is accomplished by drawing a Williot diagram; after the strain in each member has been determined, as described in Sec. I. In order to compare the results of the algebraic method with the graphic, the latter will be explained in connection with the problem illustrated in Fig. 176. Since the modulus of elasticity is constant for all members, the quantities in column 4 of Table I will be taken to represent the strains; and the factors 1,000 and $E = 29{,}000{,}000$ lb. per

square inch will be introduced at the end of the problem, as in the algebraic solution. Three solutions will be given, based on three different assumptions.

First Solution.—We shall assume that the point 6 is fixed in position and the member 5–6 fixed in direction; and determine the relative displacements of the other points, with reference to point 6, when the various members are subject to the strains, as indicated in column 4, Table I. First let us consider the triangular unit 4–5–6 of the truss in Fig. 176. For convenience this unit is shown in Fig. 180, and designated as *abc*. From column 4 the strains are represented by the following quanti-

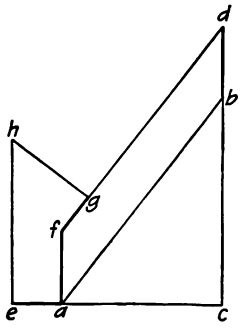


FIG. 180.

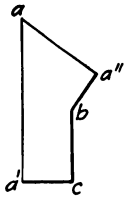


FIG. 181.

ties, $bc = +4{,}610$; $ac = +3{,}150$ and $ab = -2{,}680$. According to the hypothesis, the point c is to remain fixed in position and the member bc is to remain fixed in direction.

The strain in bc is an increase in length, represented by the quantity 4,610; consequently from b we lay off to any convenient scale (not necessarily the scale used in laying off bc) $bd = 4{,}610$, above the point b. The point which was originally at b has now moved to d. The location of d was a simple matter; since according to the hypothesis, the member bc (when elongating) had no option except to extend vertically upward.

The location of the new position of the point at a is more complicated. The member ac lengthens; and since c is fixed, the extension is to the left of and away from c. Hence we lay off $ae = 3{,}150$. The movement of a is also influenced by the member ab. Hence af is laid off equal and parallel to bd, and fd is

drawn to represent the original length of *ab*. The member *ab* is shortened by the strain 2,680; since the point *d* has been located; therefore, the point *f* moves towards *d*, the amount *fg*

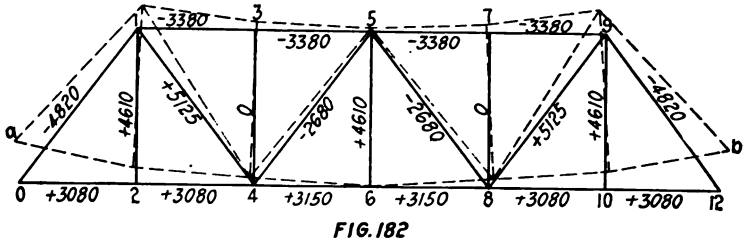


FIG. 182

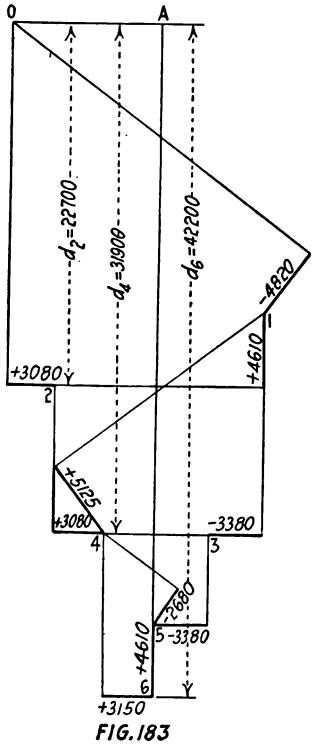


FIG. 183

= 2,680. Since the two members *ac* and *ab* are connected; the two points *e* and *g*, which represent the ends of these members, when under strain, must coincide. Therefore an arc having the radius *ce* is described about *c* as a center, and an arc having

the radius dg is described about d as a center. The intersection of the two arcs at h marks the new position of the point, which was originally at a. Since, in any practical problem, the arcs are very small in proportion to their radii; it will be sufficiently accurate to draw the straight lines eh and gh perpendicular to ac and ab, respectively.

Reference to Fig. 181 shows at once that the displacement of the points a and b may be determined, without including in the diagram the members themselves. Let the point c, which is assumed fixed in position be the origin or fixed point of reference. The member cb lengthens, and the point b moves upward and away from c; therefore lay off $cb = 4{,}610$, upward from c. The member ca also lengthens, and the point a moves horizontally to the left from c; therefore lay off $ca' = 3{,}150$, to the left from c. The member ab shortens, and the point a moves (also) diagonally upward and to the right towards b; therefore lay off $ba'' = 2{,}680$, parallel to ab upward and to the right from b. The perpendiculars through a' and a'' intersect at a. Since c is the origin or reference point, the movements or displacements of the points a and b from their *original* position are indicated by their positions relative to the origin c.

The procedure is similar when the frame consists of a series of triangular units, as illustrated by the truss in Fig. 182. It is assumed that the truss and loads are the same as shown in Fig. 176, and specified in Table I. The quantities which represent the strains in the various members, taken from column 4 of the table, are indicated in the figure for convenience. A positive sign indicates elongation, and a negative sign indicates a shortening. Each strain is laid off in the Williot diagram (Fig. 183) parallel to the original direction of the member in which the strain occurs. Joint 6 is assumed fixed in position, and the member 5–6 is assumed fixed in direction; hence the point 6 (Fig. 183) is the origin or reference point in the Williot diagram.

Joint 5 moves upward from joint 6, hence point 5 is laid off to a convenient scale above point 6. Joint 4 moves to the left away from joint 6, and upward to the right toward joint 5; hence the strain in 4–6 is laid off to the left from point 6, and the

strain in 4–5 is laid off upward and to the right from point 5. The perpendicular lines drawn from the extremities of these two strains intersect at point 4. The position of point 4, relative to the reference point 6, indicates the movement of point 4 from its original position.

Joint 3 moves to the right toward joint 5, and moves neither upward nor downward with reference to joint 4, since there is no strain in the member 3–4; hence the strain in 3–5 is laid off to the right from point 5, and no strain is laid off from point 4. The perpendicular lines drawn from the extremity of the strain in 3–5, and from point 4, intersect at point 3. The position of point 3, relative to point 6, indicates the movement of joint 3 from its original position.

The remaining joints 1, 2 and 0 are similarly treated in consecutive order; and the joint 0 is finally located in the diagram.

The scale used in constructing the Williot diagram is too large to be of service in locating the movement or displacement of each joint with reference to the truss itself. If a smaller and more convenient scale is taken, the displacements may be indicated by the dotted outline as shown in Fig. 182; although, even there, the actual distortion is greatly exaggerated in comparison to the dimensions of the truss. On account of symmetry in loading and design, the distortion of the whole truss may be determined by drawing the Williot diagram for either half. The dotted outline gives a true conception of the distortion, when joint 6 is fixed in position, the member 5–6 fixed in direction, and the members are subjected to the strains as indicated. It is evident, however, that joints 0 and 12, at the points of support instead of joint 6, remain fixed in elevation. The correction for this error in assumption is made by moving the dotted outline vertically downward until the joints a and b are at the same elevation as 0 and 12. The corresponding correction is made in the Williot diagram by considering A as the origin, or reference point, instead of point 6. The vertical displacements or deflections of joints 2, 4 and 6 are obtained by scaling d_2, d_4 and d_6; multiplying each by 1,000 and dividing by 29,000,000; as in the algebraic solution.

The horizontal displacement of any point is obtained by scaling its horizontal distance from the line $A6$. Thus the

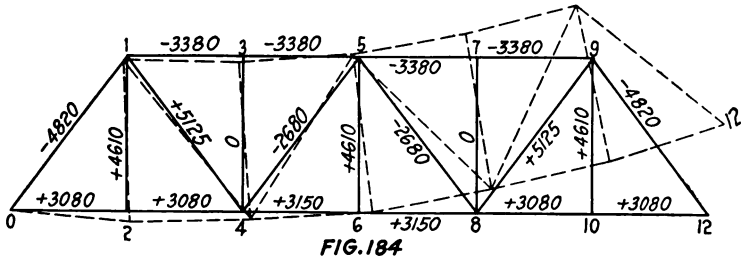


FIG. 184

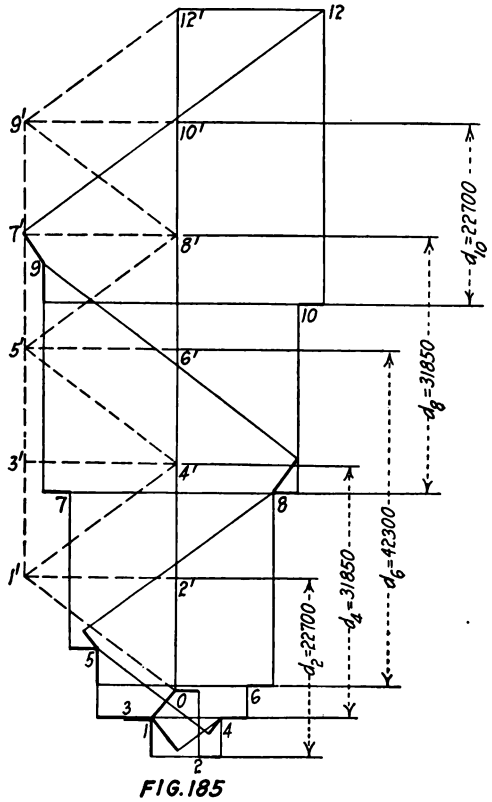


FIG. 185

horizontal, as well as vertical displacements of all joints in the truss, may be found by drawing one Williot diagram.

Second Solution.—If the loading or design in the first solution

had not been symmetrical, the assumption that the member 5–6 remained fixed in direction would not have been true; and a correction would have been necessary. The method by which this correction may be made will be illustrated by using the same strains as before, and assuming that the joint o (Fig. 184) remains fixed in position, and the member o–1 fixed in direction. The Williot diagram is shown in Fig. 185, where the point o is the origin or reference point. The point 1 is located first; then 2, 4, 3, 5 and so on to point 12.

The distorted outline may now be drawn, as shown by the dotted lines in Fig. 184. This outline has precisely the same configuration as the distorted outline of Fig. 182. In order that the distorted outline may truly represent the deflections of the various joints, it is necessary that the dotted figure be rotated about joint o; until the joint 12 of the dotted outline is on a level with joint 12 of the original outline. During this rotation, the vertical movement of joint 12 of the dotted outline is represented by the vertical distance from point 12 to point o in the Williot diagram. Let r represent this vertical distance. Draw the horizontal line 12–12′, intersecting the vertical line o–12′ at 12′. Then r is the distance o–12′. Taking o–12′ as the bottom chord, draw the truss to scale as shown by dotted lines in Fig. 185. This dotted outline is called the Mohr rotation diagram, which supplements the Williot distortion diagram. The total vertical displacement or deflection of any joint may be considered as the result of two operations—distortion and rotation. As the distortion takes place, joint 2 is lowered the vertical distance from point o to point 2 in the distortion diagram. As the distorted outline is rotated about joint o, and joint 12 moves through the vertical distance r, joint 2 is lowered the additional distance $\frac{1}{6}r$, represented by the distance 2′–o in the rotation diagram. Hence the total deflection of joint 2 is represented by the distance d_2, which scales 22,700.

Joint 10 is raised by distortion the vertical distance from point o to point 10; and lowered by rotation $\frac{5}{6}r$ or the distance 10′–o. Hence the net deflection of joint 10 is represented by the distance d_{10}, which also scales 22,700. Similarly the vertical displacement of joint 8 is represented by the vertical distance

from point 8′ in the rotation diagram, to point 8 in the distortion diagram; and the horizontal displacement of joint 9 is the horizontal distance from point 9′ in the rotation diagram, to point 9 in the distortion diagram.

It will be seen by inspection that the *vertical* displacements of all joints as obtained from Fig. 185 are the same as given in Fig. 183, or by the algebraic solution in Table I. The horizontal displacements differ, because in Fig. 183 the truss was held at joint 6 instead of at joint 0, as in Fig. 185.

Third Solution.—In the first solution, a translation of the distorted outline (Fig. 182) was necessary, because of the erroneous assumption of a joint fixed in position; while in the second solution, a rotation of the distorted outline (Fig. 184) was necessary, because of the erroneous assumption that a member was fixed in direction. The Williot distortion diagram, in Fig. 187, is drawn on the assumption that the joint 4 is fixed in position and the member 4–1 is fixed in direction. Point 4 is the origin; and the other points may be located in the order 1, 2, 0, 3, 5, 6 and so on to 12. The distorted outline of the truss, shown in Fig. 186, has precisely the same configuration as in Figs. 182 and 184. The Mohr rotation diagram will be drawn on the basis of a fixed support at 0, and a roller support at 12. It is apparent that the distorted outline must be translated vertically downward, then horizontally to the right; until joint 0 of the distorted outline coincides with joint 0 of the original outline. The distorted outline must then be rotated about joint 0, until joint 12 of the distorted outline is level with joint 12 of the original outline. Since joint 0 is now to be considered fixed in position, instead of joint 4, the origin is moved from point 4 to point 0 in Fig. 187. Draw the vertical through point 0, and the horizontal through point 12, intersecting at 12′; and on 0–12′, as the bottom chord, construct the truss to scale.

The total vertical displacement or deflection of any joint may be considered as the result of three operations—distortion, vertical translation and rotation. As the distortion takes place, joint 2 is lowered the vertical distance from point 4 to point 2 in the distortion diagram. The vertical translation lowers the

joint the vertical distance from point o to point 4. Finally the rotation lowers the joint the vertical distance from point 2′ to

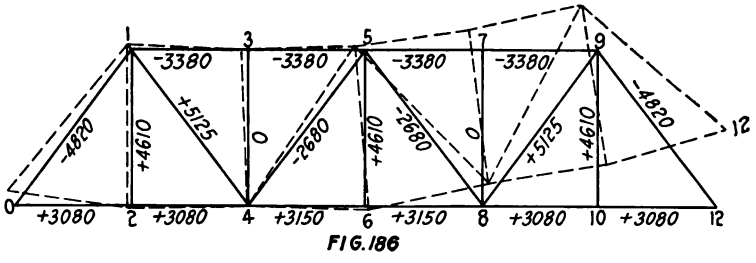


FIG. 186

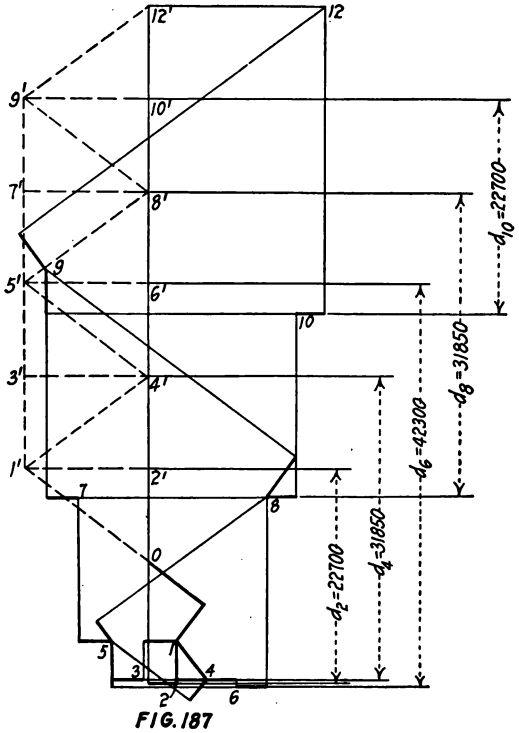


FIG. 187

point o. Hence, the total vertical distance through which the joint 2 moves, is represented by the vertical distance from point 2′ in the translation rotation diagram to point 2 in the distortion diagram; which is $d_2 = 22{,}700$. The vertical displacements or

deflections here found agree with those of the previous solutions. The horizontal displacements agree only with those of the second solution, where joint o was also subject to horizontal restraint

SEC. III. GENERAL CONSIDERATIONS

Deflections are the result of changes in length of the members or distances between panel points; whether caused by strains, temperature, or play between the pins and pin-holes. The treatment in all cases is the same after the changes in length have been determined.

182. Temperature.—Suppose that we wish to determine vertical displacement of the point 4, Fig. 176; when the temperature of the top chord is 10 degrees above normal and the temperature of the bottom chord is 15 degrees below normal. The coefficient of thermal expansion is 0.0000065. Consider the member 1–3. Its change in length, due to 10 degrees rise in temperature, is

$$324 \times 10 \times 0.0000065 = +0.021 \text{ in.}$$

This change in length is treated precisely as if it were a strain resulting from the stress in the member, due to loads. Since the coefficient of thermal expansion is the same for all members,

TABLE II

Member	Length in inches l	Temperature change t	lt	u_4	ltu_4
1–3	324	+10	+3,240	−1.0	−3,240
3–5	324	+10	+3,240	−1.0	−3,240
5–7	324	+10	+3,240	−0.5	−1,620
7–9	324	+10	+3,240	−0.5	−1,620
o–2	324	−15	−4,860	+0.5	−2,430
2–4	324	−15	−4,860	+0.5	−2,430
4–6	324	−15	−4,860	+0.75	−3,645
6–8	324	−15	−4,860	+0.75	−3,645
8–10	324	−15	−4,860	+0.25	−1,215
10–12	324	−15	−4,860	+0.25	−1,215

$\Sigma ltu_4 = -24{,}300$

the computation may be arranged as in Table II. The vertical displacement of point 4 is

$$\Delta_4 = \Sigma ltu_4 \times 0.0000065 = -0.16 \text{ in.}$$

The negative sign indicates that the point 4 is raised by the effect of the temperature.

The problem may be solved graphically by drawing a Williot diagram and using the quantities *lt* in Table II, to represent the changes in length.

183. Camber.—It is desirable that the truss in Fig. 175, when loaded, shall have the configuration represented by the full lines, rather than by the dotted lines. In order to accomplish

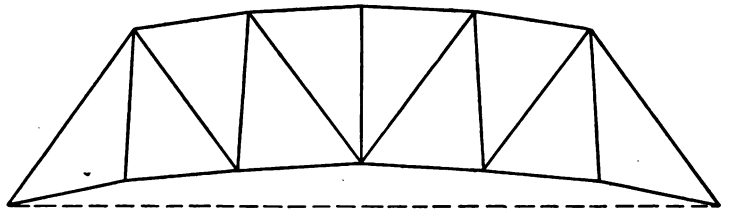


FIG. 188.

this end, it is necessary to *camber* the truss, by increasing the length of each compression member and decreasing the length of each tension member by the amount of strain which it experiences under load. Thus, in Fig. 176, the strain in the member 1-3 from Table I is

$$\frac{-3{,}380 \times 1{,}000}{29{,}000{,}000} = -0.116 \text{ in.}$$

This member is shortened about one-eighth of an inch when the truss is loaded, hence its original length is made 27′–0⅛″. If all members are treated accordingly, and the truss erected on false work, in such a way that the members are carrying little or no stress; the truss will have the configuration of Fig. 188; which is known as a camber diagram. The camber diagram is constructed from a Williot diagram, drawn by using the strains as given in column 4 of Table I, with opposite signs. Trusses are usually cambered for dead load plus live load, impact not included; sometimes the dead load plus two-thirds the live load is taken.

The following approximate method is sometimes used. If the average unit stress in the members is 14,000 lb. per square inch, based on the gross section; the strain in every 10 ft. of length is a little short of one-sixteenth of an inch. Hence a truss may be cambered by a rule-of-thumb method, if one-eighth of an inch for every 10 ft. in length is added to the top chord; the length of all other members remaining the same as if no allowance were made. Thus each top chord member of the truss in Fig. 176 would be made $27'-0\frac{5}{16}''$ long.

184. Maxwell's Theorem of Reciprocal Deflection.—The proof of Maxwell's theorem as applied to beams, on page 231, is

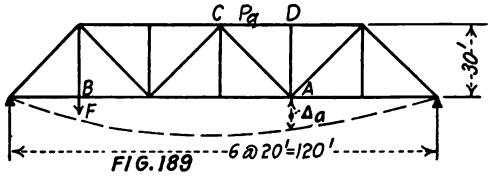


FIG. 189

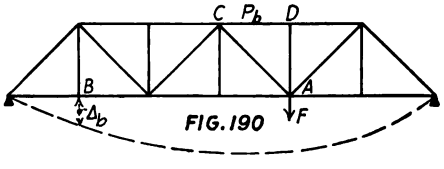


FIG. 190

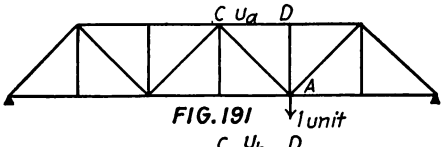


FIG. 191

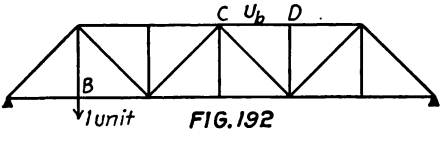


FIG. 192

equally applicable to trusses. It is also easily proven by the algebraic method of Sec. I. Let Δ_a represent the deflection at A, caused by the load F at B (Fig. 189); and let Δ_b represent the deflection at B, caused by the load F at A (Fig. 191); then

$$\Delta_a = \Sigma \frac{P_a u_a l}{AE}$$

and

$$\Delta_b = \Sigma \frac{P_b u_b l}{AE}$$

where P_a = stress in any member of Fig. 189.
P_b = stress in any member of Fig. 190.
u_a = stress in any member of Fig. 191.
u_b = stress in any member of Fig. 192.

Choose any member as CD, then

$$P_a = \frac{2F}{9}, P_b = \frac{8F}{9}, u_a = \frac{8}{9} \text{ and } u_b = \frac{2}{9}$$

hence $P_a u_a = P_b u_b$

If any other member be taken at random it will be found that $P_a u_a = P_b u_b$

therefore $\Delta_a = \Delta_b$

or the deflection at A, caused by a load at B, equals the deflection at B caused by the same load at A.

CHAPTER VIII

SWING BRIDGES

A swing bridge rotates in a horizontal plane about a vertical axis, and is classified as *center-bearing* or *rim-bearing* in accordance with the method by which it is supported while swinging.

SEC. I. CENTER-BEARING SWING BRIDGES

185. General Considerations.—When a bridge of this type is open, each truss is supported at the end of a cross-girder which rests on a center-bearing c (Fig. 193). This bearing is usually a phosphor-bronze disk, from 1 to 3 ft. in diameter, between two hardened steel disks. To prevent the bridge from tipping; balance wheels w, about 18 in. in diameter and six to eight in number, are fastened to the trusses and floor system in such a way that they roll on a circular track t. These wheels are so adjusted that they carry no load, except when the bridge is out of balance on account of wind pressure or similar causes.

When the bridge is closed; the ends a, b, d and e are raised a proper amount by wedges. Wedges are also inserted at f and g a sufficient amount to give the trusses a bearing on the pier independent of the cross-girder; but no attempt is made to lift the trusses at these points in order to relieve the cross-girder of any of the dead load. Thus at the center support, the dead load is carried by the cross-girder, while the live load is supported directly by the pier.

186. Conditions of Loading.—When the bridge is open, the dead load is balanced about the center support. When the bridge is closed, the total dead load reaction at the center is relieved, as the wedges are driven and the ends raised. (If the ends were raised to a sufficient height, the bridge would be lifted free of the center support; and the reaction which formerly supported the bridge would be transferred from the center

to the ends.) The mechanical parts of the bridge are usually designed in such a way that the end wedges, when fully driven, will provide a positive dead-load reaction; somewhat greater than the negative live-load and impact reaction. This insures the bridge against hammering of one end when the train comes on at the other end. By this arrangement, the dead-load end reactions are less than those obtained theoretically when the truss is considered as a continuous beam over three level supports. The cost of installation and operation are thereby reduced.

If hammering is eliminated and the ends remain fixed in

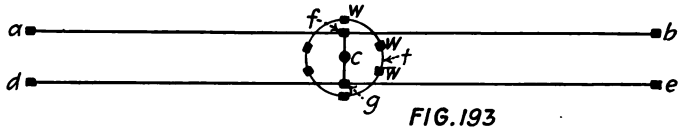


FIG. 193

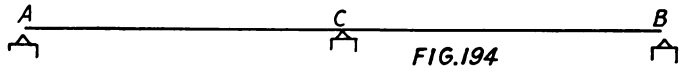


FIG. 194

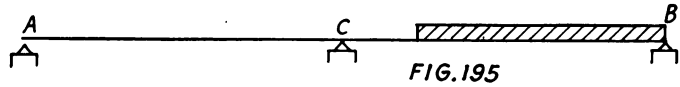


FIG. 195

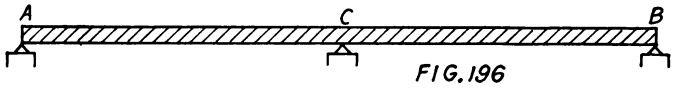


FIG. 196

elevation, the live-load reactions may be computed on the basis of complete continuity of each truss. If, however, on account of error in design or adjustment of the wedges; or because of settlement of the piers, it happens that no dead-load end reaction is present, we have the condition as illustrated in Fig. 194. Let us assume, for example, that a very small clearance exists between the beam and its end supports. The total weight of the beam is supported at C, as is the case when the bridge is open. When the live load comes on the arm BC (Fig. 195), the beam tilts until it finds a bearing at B; and the live load is supported by BC acting as a simple span. However, the dead load of weight of the beam is still entirely supported

at C. If the live load is present in both arms, (Fig. 196) the beam will deflect until it has a bearing at A and B; and thus a condition of continuity, or partial continuity, must be considered in finding the live load reactions. The extent of the continuity will depend upon the amount of clearance which previously existed at A and B in Fig. 194. The dead load is still totally supported at C. In order to provide for these contingencies, the following conditions or classes of loading should be considered.

Case I.—Dead load; bridge open or wholly supported at the center.

Case II.—Dead load; bridge closed, both ends raised until a definite end reaction is attained; amount to be specified after negative live load and impact end reaction have been determined.

Case III.—Live load on one arm only; acting as a simple span.

Case IV.—Live load on one arm only; continuous girder action.

Case V.—Live load on both arms; continuous girder action.

If the ends are raised we have Case II, combined with Case IV or Case V. If the ends are not raised we have Case I, combined with Case III or Case V.

The bottom chords are usually better protected from the rays of the sun than the top chords. Because of this, and also on account of cold weather and ice, it is apparent that the temperature of the top chord may be considerably higher than that of the bottom chord. Thus the top chord may be lengthened and the bottom chord shortened, causing the span to "hump" at the center; thereby relieving the center reactions somewhat and increasing those at the ends.

Except in special cases the temperature factor is neglected.

187. Stresses in a Swing Span.—The stresses for the five cases of loading will be computed for the 300-ft. span (Fig. 197). The assumed dead load is 3,000 lb. per foot of span, or 37,500 lb. per panel per truss. The end panel load is assumed at 20,000 lb. A Cooper's E-40 will be taken for the live load. The impact allowance will be computed from the formula

$$I = L\frac{300}{l + 300}$$

TABLE I

Member	Dead load		Live load						Combinations				
			Case III		Case IV			Case V					
					Left arm loaded		Right arm	Both arms loaded					
	Case I	Case II 30,000 lb. uplift net	Left segment	Right segment	Left segment	Right segment		Broken	Continuous	I and III		II and IV or V		I and V
L_0U_1	+26	−39	−211 −141 −352		−180 −120 −300		+30 +20 +50			−352 + 18 −334	+26	−300 − 39 −339	+50 −26 +24	
U_1U_2	+65	−19	−208 −139 −347		−170 −113 −283		+38 +25 +63			−347 + 44 −303	+65	−283 − 19 −302	+63 −12 +51	
U_2U_4	+254	+87	−208 −139 −347		−132 − 88 −220		+ 76 + 50 +126			−347 +169 −178	+254	−220 + 58 −162	+126 + 87 +213	
U_3L_3	−271	−205	−211 −141 −352		−241 −161 −402		−30 −20 −50	−271	−250 −125 −375	−352 −271 −623		−402 −205 −607		−375 −271 −646
U_3U_4	+569	+319	0		+115 + 77 +192		+115 + 77 +192	+230	+206 +103 +309		+569	+309 +319 +628		+309 +569 +878
L_0L_2	−17	+25	+135 + 90 +225		+115 + 77 +192		−19 −13 −32			+225 − 12 +213	−17	+192 + 25 +217	−32 +16 −10	

L_2L_4	−144	−19	+235 +157 +392		+178 +119 +297		− 57 − 38 − 95			+392 − 96 −296		+297 − 13 +284	− 95 − 19 −114	
L_4L_6	−396	−187	+135 + 90 +225		−4 −3 −7	+42 +34 +76	− 95 − 63 −158	−99			−396	−158 −187 −345		− 99 −396 −495
U_1L_2	−75	−10	−11 −10 −21	+140 +100 +240	−14 −13 −27	+113 + 81 +194	−30 −20 −50	−44		+240 − 50 +190	−21 −75 −96	+194 − 7 +187	−50 −10 −60	− 44 − 75 −119
L_2U_3	+124	+59	+39 +33 +72	−82 −63 −145	+48 +41 +89	−61 −47 −108	+30 +20 +50	+78		+72 +124 +196	−145 +82 −63	−108 + 40 − 68	+ 89 + 59 +148	+ 78 +124 +202
U_3L_4	−172	−107	− 82 − 63 −145	+39 +33 +72	− 99 − 76 −175	+27 +23 +50	−30 −20 −50	−129	−106 − 53 −159	−145 −172 −317		−175 −107 −282		−159 −172 −331
L_4U_5	+221	+156	+140 +100 +240	−11 −10 −21	+165 +118 +283	− 7 − 5 −12	+30 +20 +50	+195	+185 + 98 +283	+240 +221 +461		+283 +156 +439		+283 +221 +504
U_1L_1	+38	+38	+ 76 + 65 +141		+ 76 + 65 +141					+141 + 38 +179				

in which I = impact stress
L = live load stress
l = loaded length of span causing the live-load stress.

Since the bridge is symmetrical about the center, the stresses will be computed for the left arm only. In Case III, this arm is considered as a span simply supported at L_0 and L_6. The stresses are computed in the ordinary way. The results expressed in 1,000 lb. units are tabulated in Table I; the impact stresses appearing directly below the corresponding live-load stresses. Thus for the member U_1U_3 the live-load stress is $-208{,}000$ lb. and the impact stress is $-139{,}000$ lb. For the member L_2U_3, the live-load and impact stress is $+72{,}000$ lb. when the left segment of the arm is loaded; and $-145{,}000$ lb. when the right segment of the arm is loaded.

In Cases IV and V the reactions, being statically indeterminate, cannot be accurately computed until the truss has been designed. In order that a preliminary design may be made; the reactions will be tentatively determined by assuming that the truss functions as a beam of uniform cross-section, continuous over three level supports. Only the end reactions R_0 are necessary. They are given in Table II as determined from Eqs. (9) and (11) page 257. By this process it is possible to compute the stresses and make a preliminary design, after which the true reactions may be determined.

TALBE II

1 lb. at	R_0	1,000 lb. at	R_0
L_1	+0.793	L_7	−0.064
L_2	+0.593	L_8	−0.092
L_3	+0.406	L_9	−0.094
L_4	+0.241	L_{10}	−0.074
L_5	+0.103	L_{11}	−0.040

188. Positive Shear in Panel 0-1. Case III.—The influence line *abc* for shear in the panel is shown in Fig. 197, from which the following criterion for maximum positive shear may be developed.

f P_1 = the load in panel 0–1
and P = the total load on the arm 0–6
the criterion is

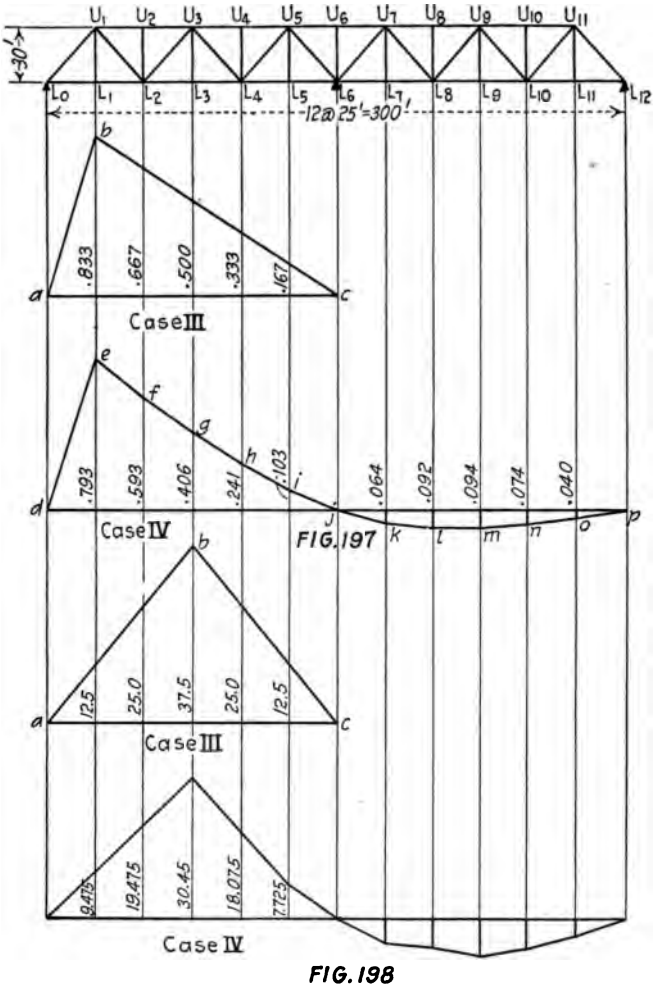


FIG. 197

FIG. 198

$$P_1 \lessgtr \frac{P}{6}$$

When the train is moving to the left, wheel 4 passing L_1 satisfies this criterion and the maximum shear in the panel is +162,000

lb. The area *abc* is 62.5 and the equivalent uniform load per linear foot is

$$q = \frac{162{,}000}{62.5} = 2{,}590$$

Case IV.—The influence line for positive shear is *defghij* (Fig. 197), from which the following criterion may be developed.

If P_1 = the load in panel 0–1
P_2 = the load in panel 1–2
P_3 = the load in panel 2–3, etc.

the criterion is

$$793P_1 \lesseqgtr 200P_2 + 187P_3 + 165P_4 + 138P_5 + 103P_6$$

Try wheel 4 at L_1, train moving to the left.

Wheel 4 approaching L_1

$793P_1 = 793 \times 50 = 39{,}650$

$200P_2 = 200 \times 66 = 13{,}200$
$187P_3 = 187 \times 56 = 10{,}472$
$165P_4 = 165 \times 73 = 12{,}045$
$138P_5 = 138 \times 57 = 7{,}866$
$103P_6 = 103 \times 50 = 5{,}150$
48,733

$39{,}650 < 48{,}733$ therefore the shear is increasing.

Wheel 4 having passed L_1

$793P_1 = 793 \times 70 = 55{,}510$

$200P_2 = 200 \times 59 = 11{,}800$
$187P_3 = 187 \times 43 = 8{,}041$
$165P_4 = 165 \times 86 = 14{,}190$
$138P_5 = 138 \times 44 = 6{,}072$
$103P_6 = 103 \times 50 = 5{,}150$
45,253

$55{,}510 > 45{,}253$, therefore the shear is decreasing.

When the train is moving to the left, wheel 4 at L_1 satisfies the criterion for maximum shear in panel 0–1.

The shear may be computed by scaling the length of the ordinate in the influence line for each load, and taking the sum of the products of each load and its ordinate. Heretofore this has been the usual method of procedure. It requires that the influence line be drawn quite accurately to scale, and considerable care taken in scaling the ordinates. The shear may

also be determined by taking the sum of each floor-beam load and its corresponding ordinate as follows:

$$\begin{aligned} 75.64 \times 0.793 &= 60.0 \\ 52.56 \times 0.593 &= 31.2 \\ 72.80 \times 0.406 &= 29.6 \\ 58.20 \times 0.241 &= 14.0 \\ 48.60 \times 0.103 &= \underline{\ 5.0} \\ 139.8 &= \text{maximum shear in panel 0–1.} \end{aligned}$$

A new and much shorter method, as outlined by the author in *Engineering News Record*, June 9, 1921, will now be explained. If the influence line *defghij* were straight from *e* to *j*, the criterion for maximum shear in panel 0–1 would be the same as for Case III. The difference between the broken line *efghij* and a straight line from *e* to *j* is so slight in this or any similar truss that the criterion for maximum shear in the panel will, in general, place the train in the same position or approximately the same position, as will the criterion for Case III. A very close approximation to the shear of 139.8 lb. can be made very quickly, by assuming the same equivalent uniform load in both cases; or, in other words, by assuming that the shears in the panel for the two cases are directly proportional to the areas of the respective influence line diagrams. These areas are proportional to the sums of their respective ordinates, thus

$$\text{area } abc : \text{area } defghij :: 2.5 : 2.14$$

The shear in the panel for Case III was found to be 162 lb., hence the shear for Case IV is

$$162. \times \frac{2.14}{2.5} = 138.3$$

This is a reasonably close approximation to the actual shear of 139.8 lb., previously determined.

It is now obvious that the stresses in L_0U_1 and L_0L_2 resulting from positive shear, may be quickly found by multiplying the stresses for Case III by the ratio 2.14/2.5. This ratio for panel 0–1 remains the same for any bridge having six equal panels in each arm, irrespective of the length of the panel.

The stresses for Case III are given in Table I; and the live load stresses in L_0U_1 and L_0L_2 for Case IV are as follows:

$$L_0U_1 = -211 \times \frac{2.14}{2.5} = -180$$

$$L_0L_2 = +135 \times \frac{2.14}{2.5} = +115$$

The impact stresses for Case IV are determined in a similar manner.

The influence line for negative shear in panel 0–1 for Case IV is *jklmnop*, and since there is at present no corresponding area for Case III we shall leave this question to be considered later.

189. Positive Moment about U_3. Case III.—The influence line *abc* for moment about U_3 is shown in Fig. 198.

Let P_1 = the load on the segment 0–3
and P = the total load on the area 0–6

then the criterion for maximum moment about U_3 is

$$P_1 \lessgtr \frac{P}{2}$$

Wheel 12 at L_3 satisfies this criterion, and the maximum moment about U_3 is 7,057 ft.-lb.; hence the maximum live load stress in L_2L_4 is

$$\frac{7{,}057}{30} = +235$$

Case IV.—The influence line is *defghij*, Fig. 198.

If P_1 = the load in panel 0–1
P_2 = the load in panel 1–2
P_3 = the load in panel 2–3, etc.

the criterion for the maximum moment about U_3, when reduced, is

$$379P_1 + 400P_2 + 439P_3 \lessgtr 495P_4 + 414P_5 + 309P_6.$$

Upon trial it will be found that wheel 12 at L_3 satisfies this

criterion also. Multiplying each floor-beam load by its corresponding ordinate,

$$\begin{aligned} 75.20 \times\ 9.475 &= \ 713 \\ 51.92 \times 19.475 &= 1{,}011 \\ 74.52 \times 30.450 &= 2{,}269 \\ 56.56 \times 18.075 &= 1{,}022 \\ 48.80 \times\ 7.725 &= \ 377 \\ & 5{,}392 = \text{maximum moment about } U_3. \end{aligned}$$

The maximum tensile stress in L_2L_4 is

$$\frac{5{,}392}{30} = +179.7$$

This value may be closely approximated from Case III as follows: The ratio of the sum of the ordinates in Case IV, to the sum of the ordinates in Case III, is

$$\frac{85.2}{112.5} = 0.756$$

and $$235 \times 0.756 = +178 = \text{stress in } L_2L_4.$$

190. Negative Shear in Panel 0-1. Case IV.—We are now prepared to consider the negative shear in panel 0–1, when the right arm 6–12 is loaded. The influence line is *jklmnop* (Fig. 197), and the criterion developed therefrom is

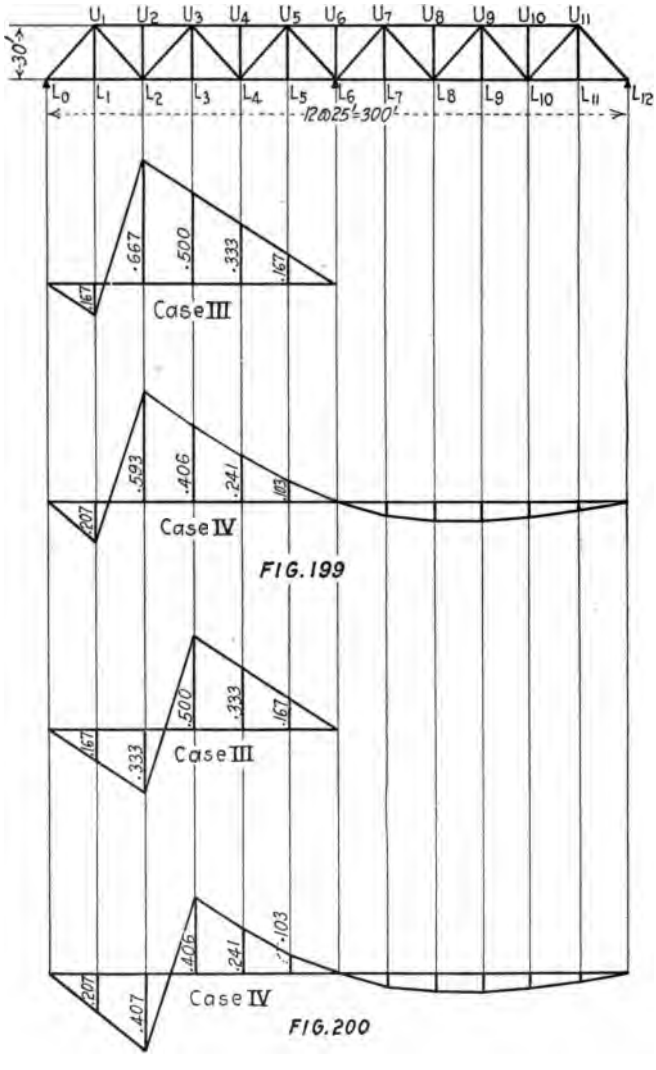
$$64P_7 + 28P_8 + 2P_9 \lessgtr 20P_{10} + 34P_{11} + 40P_{12}.$$

There are two positions of the train, when moving to the left, which satisfy this criterion; namely, wheel 11 at L_2 and wheel 8 at L_8. The maximum shear, occurring when the train is in the latter position, is -22.8 lb.; as found by taking the sum of the products of each floor beam load and its corresponding ordinate. This value may be closely approximated by the proportionate method, as follows: Since the influence line *jklmnop* (Fig. 197) has its longest ordinate at the center of the arm, it will be compared with the influence line *abc* (Fig. 198), which also has its longest ordinate at the center of the arm. The latter influence line is for the moment at U_3 for Case III, and the moment is 7,057. The ratio of the sum of the ordinates

jklmnop (Fig. 197) to the sum of the ordinate *abc* (Fig. 198) is

$$\frac{-0.364}{112.5} = -0.00324$$

and $7{,}057 \times -0.00324 = -22.9 =$ shear in panel 0–1.



This quantity represents also the maximum negative reaction at L_0 when the right arm is loaded, from which all the stresses

may be determined as given in the designated column of Table I.

191. Shear in Panel 1-2.—The influence lines are shown in Fig. 199. The stresses in U_1L_2 for Case III, as given in Table I, are -11 and $+140$; from which the stresses for Case IV may be found as follows:

$$-11 \times \frac{0.207}{0.167} = -14$$

$$+140 \times \frac{0.593 + 0.406 + 0.241 + 0.103}{0.667 + 0.500 + 0.333 + 0.167} = +113$$

The stresses for Case IV, determined by the exact method, are -15 and $+113$ respectively.

192. Shear in Panel 2-3.—The influence lines are shown in Fig. 200. The stresses in L_2U_3 for Case IV, determined by the proportionate method are

$$+39 \times \frac{0.207 + 0.407}{0.167 + 0.333} = +48$$

$$-82 \times \frac{0.406 + 0.241 + 0.103}{0.500 + 0.333 + 0.167} = -61$$

The stresses for Case IV, found by the exact method, are $+48$ and -60 respectively.

193. Shear in Panel 3-4.—The influence lines are shown in Fig. 201. The stresses in U_3L_4 for Case IV, determined by the proportionate method, are

$$-82 \times \frac{0.207 + 0.407 + 0.594}{0.167 + 0.333 + 0.500} = -99$$

$$+39 \times \frac{0.241 + 0.103}{0.333 + 0.167} = +27$$

The stresses for Case IV, determined by the exact method are -101 and $+26$ respectively.

194. Shear in Panel 4-5.—The influence lines are shown in Fig. 202. The stresses in L_4U_5 for Case IV, determined by the proportionate method, are

$$+140 \times \frac{0.207 + 0.407 + 0.594 + 0.759}{0.167 + 0.333 + 0.500 + 0.667} = +165$$

$$-11 \times \frac{0.103}{0.167} = -7$$

The stresses for Case IV, determined by the exact method, are +167 and −5 respectively.

195. Shear in Panel 5-6.—The influence lines are shown in

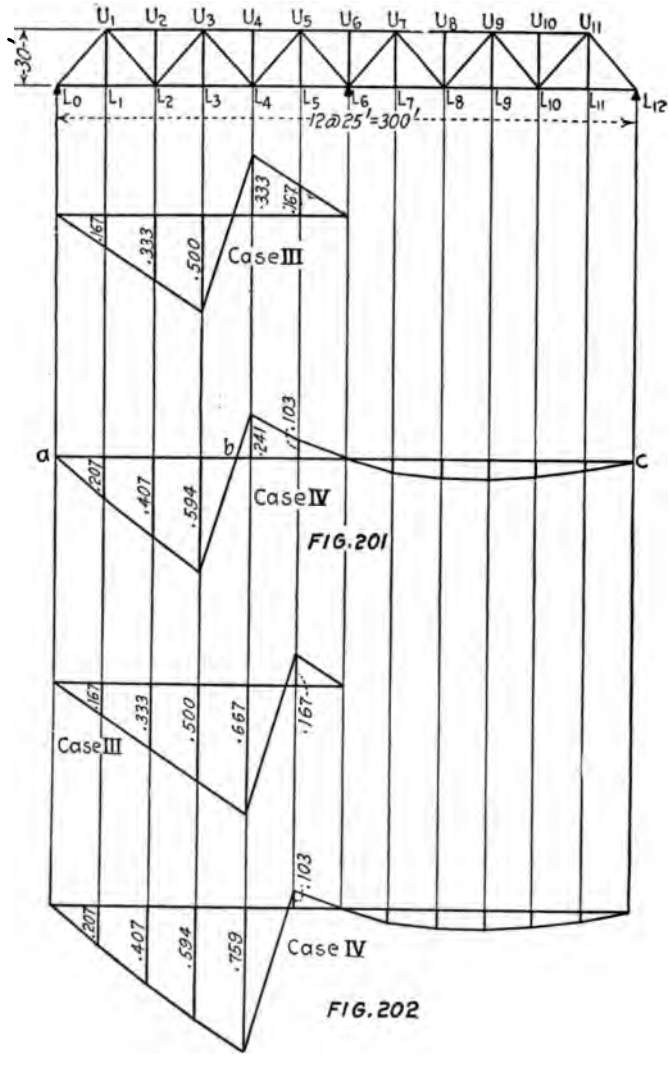


FIG. 201

FIG. 202

Fig. 203. The stress in U_5L_6 for Case IV, determined by the proportionate method, is

$$-211 \times \frac{0.207 + 0.407 + 0.594 + 0.759 + 0.897}{0.167 + 0.333 + 0.500 + 0.667 + 0.833} = -241$$

The stress for Case IV, determined by the exact method, is -241.

196. Moment about L_2.—The influence lines are shown in

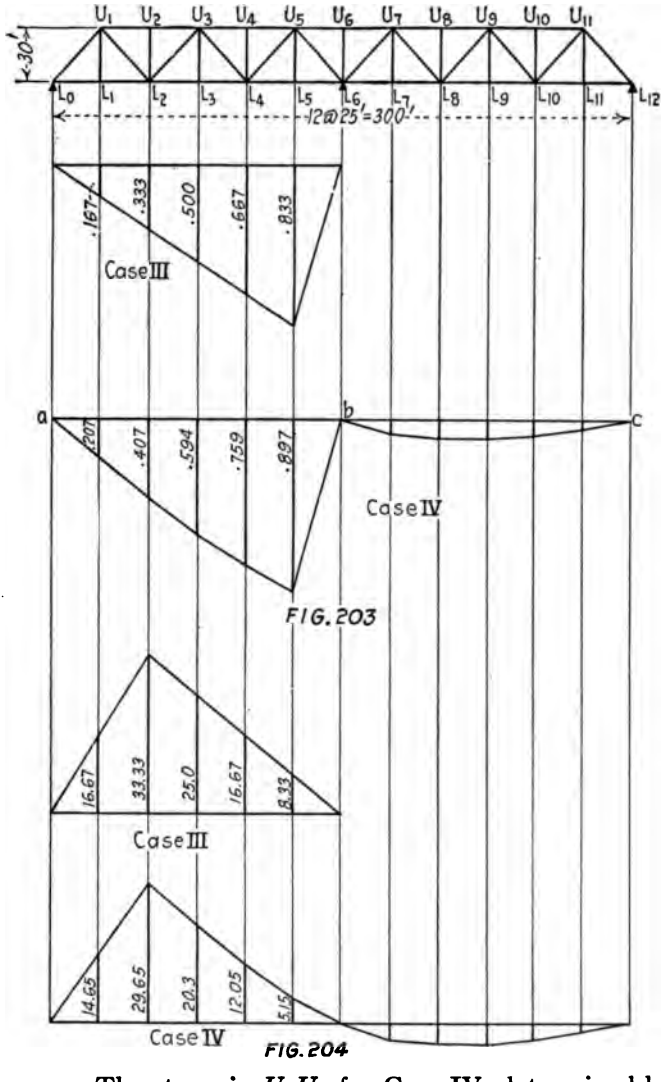


Fig. 204. The stress in U_1U_3 for Case IV, determined by the proportionate method, is

$$-208 \times \frac{81.8}{100} = -170$$

The stress for Case IV, determined by the exact method, is -172.

197. Moment about L_4.—The influence lines are shown in

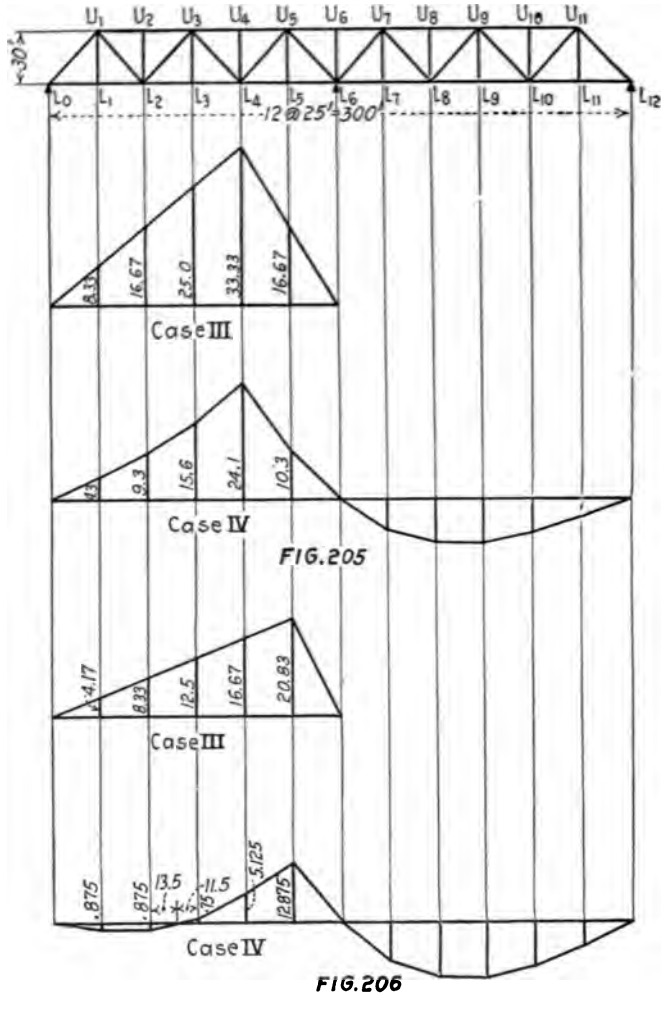


Fig. 205. The stress in U_3U_5 for Case IV, determined by the proportionate method is

$$-208 \times \frac{63.6}{100} = -132$$

The stress for Case IV, determined by the exact method, is -134.

198. Moment about U_5.—The influence lines are shown in Fig. 206. Whenever each arm has six or more equal panels, the influence line for Case IV shows a reversal of stress in one or more chord members of the loaded arm adjacent to the center support. This phenomenon is explained in connection with

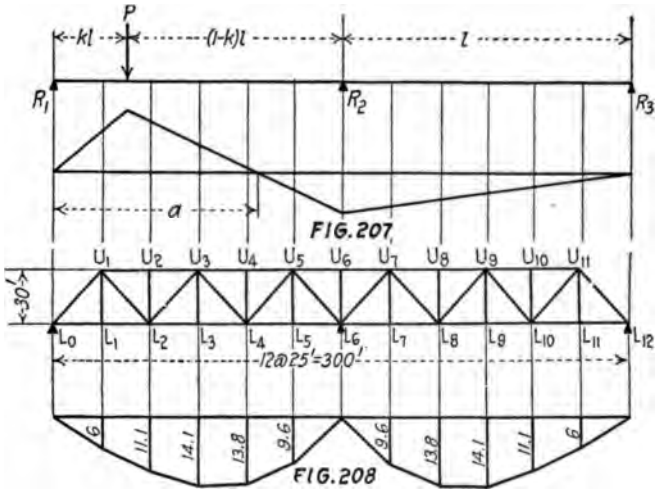


Fig. 207, where a is the distance from R_1 to the point of zero bending moment; then

$$aR_1 = P(a - kl)$$

from Eq. 9, page 257

$$R_1 = \frac{P}{4}(4 - 5k + k^3)$$

hence

$$a = \frac{4l}{5 - k^2}$$

For any position of P, the limits of k are 0 and 1; hence the limits of a are $\frac{4}{5}l$ and l. It is clear, therefore, that if any panel point experiences a negative moment from the influence load; the distance of that panel point from the center support must be less than $\frac{1}{5}$ of the arm length. If the panels are of equal length this condition can occur only when there are six or more panels in each arm.

The stresses for Case IV cannot be determined by the proportionate method as heretofore, on account of the dissimilarity of the influence line diagrams. In such instances the equivalent uniform load may be approximated by the use of equivalent uniform load table, page 206. For the right segment, $l_1 = 25$, $l_2 = 61.5$ and the equivalent uniform load is 2.7; the area is 464 and the stress in L_4L_6 is

$$\frac{464 \times 2.7}{30} = +42$$

For the left segment, it will be sufficient to call $l_1 = l_2 = 30$, and the equivalent uniform load is 2.88, hence the stress is

$$\frac{-38 \times 2.88}{30} = -4$$

199. Moment about L_6.—The influence line for Case IV is shown in Fig. 208. There is no corresponding influence line for Case III. Since the influence is symmetrical about the center, the stress in U_5U_6, when the left arm is loaded, is the same as previously determined when the right arm was loaded.

200. Case V: Both Arms Loaded. *Broken Loads.*—By referring to the influence lines for the continuous condition in Figs. 197, 198, 204 and 205, it is apparent that the stresses in the members there considered will be less for Case V than for Case IV; since any load, brought on to the right arm while the left arm is loaded, will decrease the stress because the influence areas on opposite sides of the center have opposite signs. In Figs. 199, 200, 201, 202, 203, 206 and 208 the conditions are different; since loads on the right arm and the left segments of the left arm conspire, and the live-load stress in any instance is the sum of the live stresses as given for Case IV. Consider the member U_3L_4, for example, illustrated in Fig. 201. The live-load stress is -99 when the left segment of the left arm is loaded, and -30 when the right arm is loaded; hence if a train approaches on each arm, the maximum live load stress is -129, as given in the column for broken loads. No impact is added when broken loads are considered. Specifications are not usually clear on the question of broken loads. If the location of the bridge is near a large freight terminal, it is conceivable that trains might

occasionally approach simultaneously from both ends of the bridge.

Continuous Loads.—If Figs. 199, 200, 201, 202, are taken consecutively, it is apparent that the positive influence line area for Case IV is decreasing, and in Fig. 201 it becomes less than the negative area for the right arm; the difference in areas being 2.65. Therefore the stress in U_3L_4 for Case V will be greater than for Case IV.

The stress may be approximated as follows: Consider that the train moving to the left covers the whole span. Assume that the engine covers the segment *ab* and that the stress in U_3L_4, on account of the engine, is the same as in Case IV or -99 lb. The shear in panel 3–4, on account of uniform train load of 2,000 lb. per linear foot., from *b* to *c* is $-2.65 \times 2 = -5.3$ lb., and the stress in U_3L_4 is $-5.3 \times 1.3 = -6.9$. Hence the stress for Case V is $-99 - 6.9 = -105.9$. Similarly the stress in L_4U_5 is $+165 + (7.6 \times 2 \times 1.3) = +184.8$.

201. Negative Shear in Panel 5-6.—There is no positive area in the influence line for the continuous condition of Fig. 203. The train must be moving to the left, if the engines are to be on the segment *ab*; followed by a uniform train load on the segment *bc*. Since the segment *ab* is considerably longer than the length of the two engines; the shear in panel 5–6 for Cases III and IV would be considerably less, if the engines were moving toward the left instead of in the opposite direction. We shall take this difference into consideration, by computing the negative shear in panel 5–6 when the train is moving to the left. This occurs when wheel 16 is at L_5 and the shear (Case III) is -151.7 lb.; which is considerably less than -162 lb., when the train is moving to the right. Taking the same ratio of ordinates as before, the shear for Case V is

$$\begin{aligned} -151.7 \times \frac{2.864}{2.500} &= -173.8 \\ -9.1 \times 2 \quad &= \underline{-\ 18.2} \\ &\quad -192. \end{aligned}$$

The stress in U_5L_6 is $-192.0 \times 1.3 = -249.6$.

202. Moment about L_6.—The stress in U_5U_6 for Case IV was found to be $+115$, one area being loaded. When the other

arm is covered with a uniform train load of 2,000 lb. per foot, the additional stress is +91, hence the stress for U_5U_6 for Case V is +206.

203. Dead Load: Bridge Open. Case I.—The panel loads at each end are 20,000 lb. and all others 37,500 lb. Each truss is balanced on the center support and the stresses are statically determinate. They are given in Table I.

204. Dead Load: Ends Raised. Case II.—The maximum negative reaction on account of live load was found to be −22.9, to which must be added −15.3 per impact, or a total of −38.2. Hence, if the positive uplift at each end is 38.2 or greater, there will be no hammering of one end when the train covers the opposite arm of the bridge. It will be assumed that the machinery parts are to be designed and adjusted so that the end wedges, when driven, will exert an upward pressure of 50. Since there is a dead load of 20 at the end panel point, the resultant or net positive end reaction is 30; and the resulting stresses are given in Table I. If the truss were treated as fully continuous, the end reaction would be 86.4 instead of 50; and a heavier and more expensive lifting device would be required.

205. Combinations.—As previously explained, Case I is combined with either Case III or Case V; and Case II is combined with either Case IV or Case V. Only two-thirds of the dead-load stress is taken, when dead-load and live-load stresses have opposite signs. Many specifications are not clear upon the question of stress reversals. In treating reversals, each combination should be considered separately; *i.e.*, the largest positive stress of one combination should not be considered with the largest negative stress of another combination. The members have been proportioned, and the gross cross-sectional areas given in Table III.

206. Reactions from Williot Diagram.—The quantities $\frac{Pl}{A}$ in Table III, when divided by the modulus of elasticity E, are the strains in the various members when the center reaction is removed and the truss, supported at 0 and 12, carries a load of 1 lb. at joint 6 (Fig. 209.) The Williot diagram is drawn

in Fig. 210 by using the quantities $\frac{Pl}{A}$ to represent strains. The quantities d which are proportional to the deflections are

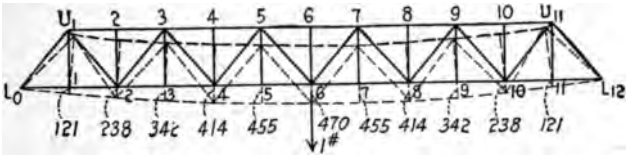


FIG. 209.

indicated in Fig. 209. The deflection at the center has been checked in Table III, where P and u_6 necessarily have the same numerical values; but it should be remembered that P is measured in pounds, while u_6 is a ratio.

It is clear from Maxwell's Theorem that if 1 lb. at joint 6 causes the deflection $d_1 = 121$ at joint 1; then 1 lb. at joint 1 will cause the deflection $d_1 = 121$ at joint 6. Hence, with 1 lb. at joint 1, the reaction at joint 6, when the truss is continuous over the three supports, is

$$R_6 = \frac{121}{470} = 0.258$$

and from statics

$$R_0 = \frac{1.0 \times 275 - (0.258 \times 150)}{300} = 0.788$$

The reactions at the left end, determined in the same manner, are given in Table IV.

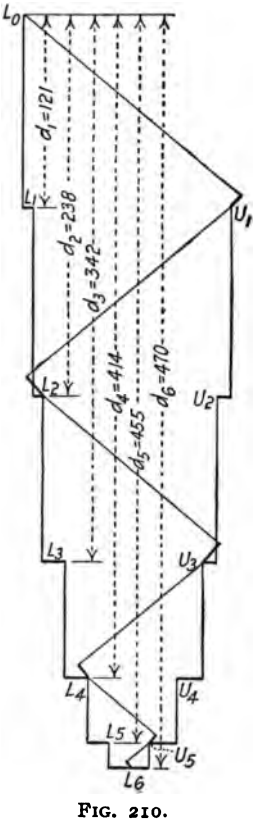


FIG. 210.

If the accurate reactions in Table IV are compared with the assumed reactions of Table II, it will be apparent that the differences are comparatively small. The greatest percentage of error occurs when the load of 1 lb. is at the center of either arm. This comparison gives a fair idea of what error may be expected, when the truss is assumed to be a beam of constant moment of inertia; and no recognition is made of deflection due to shear.

TABLE III

Member	Length, inches	Area, square inches A	Stress in pounds P	$\frac{Pl}{A}$	u_6	$\frac{Pu_6l}{A}$
L_0-U_1	469	32.5	−0.652	−9.40	−0.652	+6.13
U_1-U_2	300	30.5	−0.833	−8.20	−0.833	+6.84
U_2-U_3	300	30.5	−0.833	−8.20	−0.833	+6.84
U_3-U_4	300	30.5	−1.666	−16.40	−1.666	+27.35
U_4-U_5	300	30.5	−1.666	−16.40	−1.666	+27.35
U_5-U_6	300	68.2	−2.500	−10.99	−2.500	+27.46
L_0-L_1	300	19.8	+0.416	+6.32	+0.416	+2.63
L_1-L_2	300	19.8	+0.416	+6.32	+0.416	+2.63
L_2-L_3	300	26.5	+1.250	+14.14	+1.250	+17.68
L_3-L_4	300	26.5	+1.250	+14.14	+1.250	+17.68
L_4-L_5	300	44.5	+2.083	+14.05	+2.083	+29.25
L_5-L_6	300	44.5	+2.083	+14.05	+2.083	+29.25
U_1-L_2	469	19.8	+0.652	+15.44	+0.652	+10.08
L_2-U_3	469	19.8	−0.652	−15.44	−0.652	+10.08
U_3-L_4	469	32.5	+0.652	+9.40	+0.652	+6.13
L_4-U_5	469	44.5	−0.652	−6.88	−0.652	+4.49
U_5-L_6	469	68.2	+0.652	+4.48	+0.652	+2.92
U_1-L_1			0			
U_2-L_2			0			
U_3-L_3			0			
U_4-L_4			0			
U_5-L_5			0			
U_6-L_6			0			

$$d_6 = \Sigma \frac{Pu_6l}{A} = \frac{234.79}{2} = 469.58$$

TABLE IV

1 lb. at	R_0	1 lb. at	R_0
L_1	+0.788	L_7	−0.068
L_2	+0.581	L_8	−0.108
L_3	+0.386	L_9	−0.114
L_4	+0.226	L_{10}	−0.088
L_5	+0.099	L_{11}	−0.046

The reactions for the preliminary design might be obtained by assuming that all members have the same cross-sectional area, which may be taken as 1 sq. in.; and constructing a Williot diagram.

The combinations of stresses for Cases I and III determine the sizes of nearly all members, except the end post and chord members adjacent to the center support. Since the stresses for these cases are statically determinate, they might be used in making an estimate of the sizes of the members; and their areas used in constructing a Williot diagram.

The continuous girder formulas give such satisfactory results that a re-design is seldom necessary, except in a long span and then only for a few members adjacent to the center support.

SEC. II. RIM-BEARING SWING BRIDGES

207. The trusses in a rim-bearing swing bridge are supported by a large circular girder, which rotates with the span. The girder rests on conical rollers, usually about 18 in. in diameter; and as many rollers are used as the circumferential length of the girder will permit, in order to give as many bearings for the girder as possible, and thereby minimize the deflection. The trusses may rest directly upon the circular girder or drum, as in Fig. 211*b*. This arrangement is undesirable, because it does not give an equal distribution of the load to the rollers. A better arrangement is shown in Fig. 211*c*, where the truss loads are distributed to the drum at 8 points instead of at 4 points, as in the previous case. This arrangement gives a more even bearing. In the diagrams here shown, the center pivot receives neither dead nor live load. It is better to frame the structure so that from 15 to 20 per cent of the load is transmitted to the center pivot through radial girders. In any case each truss is supported at two points over the circular pier, as illustrated in Fig. 211*a*.

The reactions for the center-bearing bridge of the previous section were determined by assuming that the span functions as a beam of constant cross section, continuous over three supports; no allowance being made for deflection due to shear.

It was shown that the reactions thus computed by continuous girder formulas, compared very favorably with the true reactions determined after the design had been made. Such is not the case when the continuous girder formulas of Art. 167 are

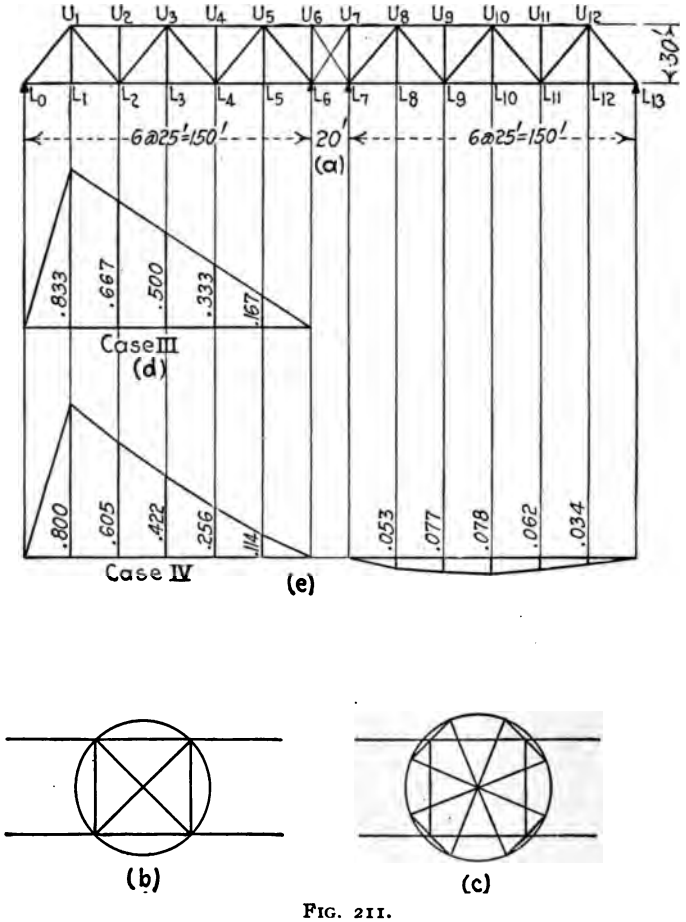


FIG. 211.

applied to a swing span on four supports. These formulas give a negative reaction at L_7 and a positive reaction at L_{13}, when the left arm is loaded. This live load negative reaction, when the impact factor is added, is in many instances, numerically greater than the dead load positive reaction; indicating that a live load over the left arm would lift the truss from its

support at L_7. This assumption is not justified either by exact analysis, made after a truss is designed; or by observed data taken after erection.

208. Partial Continuity. *Equal Moments at Center Supports.*—It will be assumed that the diagonal bracing in panel 6–7 is so light that no appreciable shear can be transmitted through this panel. The shear in the panel is assumed zero under any condition of loading; hence $R_7 = -R_{13}$ for loads on the arm 0–6, and $R_6 = -R_0$ for any loads on the arm 6–13. The formulas given in Art. 172 are applicable where $a = 20/150$; hence for an influence load of one pound

$$R_0 = (1 - k) - \frac{k - k^3}{4.8}$$

and

$$R_{13} = -\frac{k - k^3}{4.8}$$

The five cases of loading to be considered are the same as for a center-bearing bridge. The influence lines for shear in panel 0–1 for Cases III and IV are shown in Figs. 211*d* and 211*e*, respectively. If the live load is an $E40$, the stresses for Case III will be the same as for the center-bearing bridge given in Table I. The stresses for Cases IV and V may be found by the proportionate method. For example, the stress in L_0U_1 for Case III is -211, hence the stress for Case IV is

$$-211 \times \frac{.800 + .605 + .422 + .256 + .114}{.833 + .667 + .5 + .333 + .167} = -185$$

The influence lines are drawn and the stresses computed for all the members, in precisely the same manner as for the centre-bearing bridge. It may be noted that in this particular problem the positive shears are greater, and the negative shears less, in the rim-bearing type than in the center-bearing type. The stresses in some members will be greater, and in others less than in the center-bearing bridge. If, however, an independent design is made for the rim-bearing type, a comparison will show no appreciable difference in the two designs. For this reason it is a common practice to disregard the center panel when the stresses are computed. The diagonals in the center panel are

light adjustable members, which serve only to provide stability to the structure when open, and resisting a longitudinal wind pressure.

After the trusses have been designed, a sufficiently exact analysis of the reactions may be made by omitting the bracing in the center panel; removing the center supports, and drawing a Williot diagram for one pound loads placed at L_6 and L_7.

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